2 General Relativity

2.1 Homogeneous and isotropic expanding universe

2.1.1 The metric

To measure distances in spacetime we need a metric $g_{\mu\nu}$. The infinitesimal spacetime line element is then

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{21}$$

where x^μ, $\mu = 0, 1, 2, 3$, are the spacetime coordinates.

For a spatially homogeneous and isotropic universe there is a natural space time splitting given by choosing the time to be constant on the homogeneous and isotropic spatial hypersurfaces.

Note that this will cease to be the case when we introduce perturbations. Note also that the maximally symmetric spacetimes, de Sitter space, anti-de Sitter space and Minkowski space, do not have unique choices for the homogeneous and isotropic spatial hypersurfaces, and so do not have a unique natural space time splitting. Both these points will be important later on.

Let t be the proper time defined by the homogeneous and isotropic spatial hypersurfaces. Then the metric can be written in the form

$$ds^2 = dt^2 - a(t)^2 h_{ij} dx^i dx^j \tag{22}$$

where $a(t)$ is the scale factor of the universe (we will assume $a > 0$, and $\dot{a} > 0$ corresponding to an expanding universe); h_{ij} is a constant metric on the homogeneous and isotropic, but in general curved, spatial hypersurfaces; x^i, $i = 1, 2, 3$, are comoving spatial coordinates (they are called 'comoving' because they expand with the universe).

The constant comoving intrinsic curvature of the spatial hypersurfaces can be positive, negative or zero, and can be scaled to be 1, -1 or 0. The spatial metric h_{ij} can then be written in the form

$$h_{ij} dx^i dx^j = dr^2 + f^2(r)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{23}$$

where

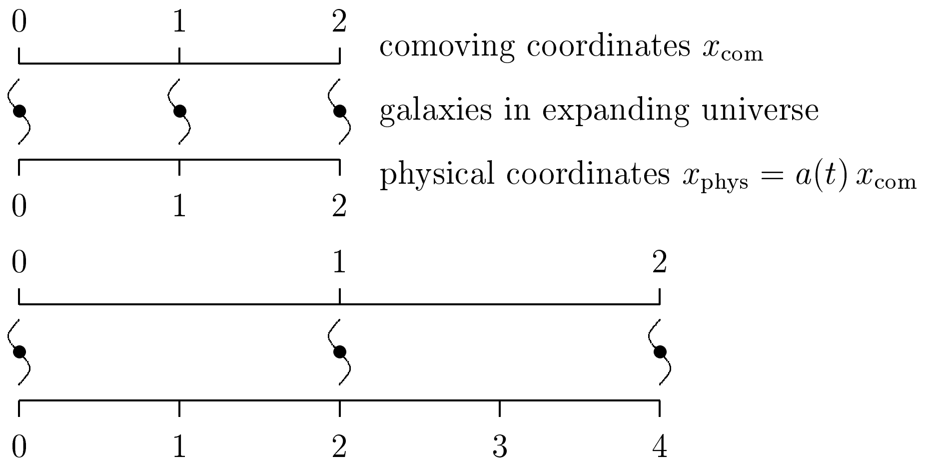
$$f(r) = \begin{cases} \sin r & \text{if} \quad K_{\rm com} = 1 \\ \sinh r & \text{if} \quad K_{\rm com} = -1 \\ r & \text{if} \quad K_{\rm com} = 0 \end{cases} \tag{24}$$

The physical intrinsic curvature of the spatial hypersurfaces K is related to the constant comoving intrinsic curvature $K_{\rm com}$ by

$$K = \frac{K_{\rm com}}{a(t)^2} \tag{25}$$

If $K = 0$ one can also write the metric in the form

$$ds^2 = dt^2 - a(t)^2\left(dx^2 + dy^2 + dz^2\right) \tag{26}$$



A physical spatial distance $x_{\rm phys}$ is related to a comoving distance $x_{\rm com}$ by

$$x_{\rm phys} = a(t)\, x_{\rm com} \tag{27}$$

For constant comoving distances

$$\dot{x}_{\rm phys} = \dot{a} x_{\rm com} = \frac{\dot{a}}{a} x_{\rm phys} \tag{28}$$

$$= H x_{\rm phys} \qquad \text{Hubble's Law} \tag{29}$$

where

$$H \equiv \frac{\dot{a}}{a} \tag{30}$$

is the Hubble parameter.

The current value of the Hubble parameter is often parametrized as

$$H_0 \equiv 100h\,{\rm km\,s^{-1}\,Mpc^{-1}} = 2.133h \times 10^{-42}\,{\rm GeV} \tag{31}$$

Observations give $h \simeq 0.65$ with an error of about 10%.

For

$$x_{\rm phys} > \frac{1}{H} = \text{Hubble distance} \tag{32}$$

we have

$$\dot{x}_{\rm phys} > 1 \tag{33}$$

so that things more than a Hubble distance away are out of physical contact and are said to be 'beyond the horizon'. Also

$$\frac{d}{dt}\left(\frac{x_{\rm phys}}{1/H}\right) = \frac{d}{dt}\left(aHx_{\rm com}\right) = \ddot{a} x_{\rm com} \tag{34}$$

for $\dot{x}_{\rm com} = 0$. Therefore, if $\ddot{a} < 0$ comoving scales move inside the horizon, and if $\ddot{a} > 0$ they move outside the horizon.

2.1.2 The Einstein equation

The dynamics of the universe is governed by the Einstein equation

$$G_{\mu\nu} = T_{\mu\nu} \tag{35}$$

For a spatially homogeneous and isotropic universe

$$T_{\mu}{}^{\nu} = \mathrm{diag}\left(\rho, -p, -p, -p\right) \tag{36}$$

where ρ is the energy density and p is the pressure (we will assume that $\rho > 0$). The Einstein equation then gives

$$3H^2 + 3K = \rho \tag{37}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p\right) \tag{38}$$

or

$$\dot{H} = -\frac{1}{2}\left(\rho + p\right) + K \tag{39}$$

The critical density ρ_{c} is defined as the energy density of a flat universe

$$\rho_{\mathrm{c}} \equiv 3H^2 \tag{40}$$

The current value of ρ_{c} is

$$\rho_{\mathrm{c}} = \left(3.000 h^{1/2} \times 10^{-3}\,\mathrm{eV}\right)^4 = 2.301 h^2 \times 10^{-120} \simeq 4\,\mathrm{GeV\,m^{-3}} \tag{41}$$

It is conventional to measure densities relative to the critical density

$$\Omega_X \equiv \frac{\rho_X}{\rho_{\mathrm{c}}} \tag{42}$$

2.1.3 Matter, fields, ...

Differentiating Eq. (37) and using Eq. (39) gives

$$\frac{d\ln\rho}{d\ln a} = -3\left(1 + \frac{p}{\rho}\right) \tag{43}$$

which is just a rewriting of $dE = -p\,dV$ and can be derived from $T^{\mu\nu}{}_{;\nu} = 0$. If

$$w \equiv \frac{p}{\rho} = \mathrm{constant} \tag{44}$$

then Eq. (43) gives

$$\rho \propto a^{-3(1+w)} \tag{45}$$

If the universe is dominated by such material, then the Einstein equation gives

$$H = \frac{2}{3(1+w)t} \tag{46}$$

and

$$a \propto t^{\frac{2}{3(1+w)}} \tag{47}$$

Now $w = \frac{1}{3}, 0, -1$ for radiation, matter, vacuum energy, respectively, so

$$\rho_{\mathrm{rad}} \propto a^{-4}\,,\ \rho_{\mathrm{mat}} \propto a^{-3}\,,\ K \propto a^{-2}\,,\ \rho_{\mathrm{vac}} = \mathrm{constant} \tag{48}$$

and for a radiation, matter, (negative) curvature, vacuum energy, respectively, dominated universe

$$H_{\mathrm{rad}} = \frac{1}{2t}\,,\ H_{\mathrm{mat}} = \frac{2}{3t}\,,\ H_K = \frac{1}{t}\,,\ H_{\mathrm{vac}} = \mathrm{constant} \tag{49}$$

and

$$a_{\mathrm{rad}} \propto t^{1/2}\,,\ a_{\mathrm{mat}} \propto t^{2/3}\,,\ a_K \propto t\,,\ a_{\mathrm{vac}} \propto e^{Ht} \tag{50}$$

The energy-momentum tensor for vacuum energy is

$$T_{\mu\nu} = \Lambda g_{\mu\nu} \tag{51}$$

and so could be put on the left hand side of the Einstein equation as a 'cosmological constant'. We will keep it on the right hand side though.

The action for a real scalar field $\phi(x^\mu)$ is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] \tag{52}$$

where g is the determinant of the metric and $g^{\mu\nu}$ is the inverse metric $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$. The energy-momentum tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi - V(\phi) \right] \tag{53}$$

Therefore, in a homogeneous and isotropic universe

$$\rho_\phi = T^0_0 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \frac{1}{a^2} \left\langle h^{ij} \partial_i \phi \, \partial_j \phi \right\rangle + V \tag{54}$$

$$p_\phi = -\frac{1}{3} T^i_i = \frac{1}{2} \dot{\phi}^2 - \frac{1}{6} \frac{1}{a^2} \left\langle h^{ij} \partial_i \phi \, \partial_j \phi \right\rangle - V \tag{55}$$

Note that any vacuum energy can be absorbed into the scalar field's potential energy $V(\phi)$.

The action for a massless vector field $A_\mu(x^\nu)$ is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right] \tag{56}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Its energy-momentum tensor is

$$T_{\mu\nu} = -g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{1}{4} g_{\mu\nu} g^{\rho\sigma} g^{\eta\kappa} F_{\rho\eta} F_{\sigma\kappa} \tag{57}$$

Therefore, in a homogeneous and isotropic universe

$$\rho_\gamma = \frac{1}{2} \frac{1}{a^2} \left\langle h^{ij} F_{0i} F_{0j} \right\rangle + \frac{1}{4} \frac{1}{a^4} \left\langle h^{ij} h^{kl} F_{ik} F_{jl} \right\rangle \tag{58}$$

$$p_\gamma = \frac{1}{3} \rho_\gamma \tag{59}$$

The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = \rho u_\mu u_\nu - p \left(g_{\mu\nu} - u_\mu u_\nu \right) \tag{60}$$

where $u^\mu = g^{\mu\nu} u_\nu$ is the fluid's four velocity, $u_\mu u^\mu = 1$.

2.2 Perturbations

In this section we will develop the general formalism for dealing with an approximately spatially homogeneous and isotropic universe. Specific applications, such as the generation of classical perturbations from quantum fluctuations during inflation, how these perturbations induce anisotropies in the cosmic microwave background radiation, and the growth of density perturbations to form galaxies and the large scale structure of the universe, will be given in later chapters. This will be one of the more technical sections.

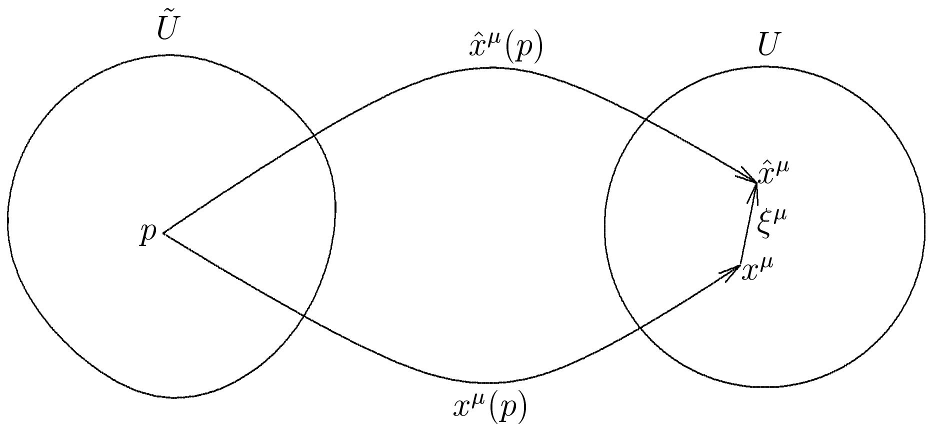
The universe, at least on large scales or at early times, is approximately spatially homogeneous and isotropic. Therefore, we can do perturbation theory using a spatially homogeneous and isotropic universe as the background.

The background universe has a natural choice of time coordinate given by taking the time to be constant on the homogeneous and isotropic spatial hypersurfaces. However, in a universe perturbed away from spatial homogeneity and isotropy, there are no homogeneous and isotropic spatial hypersurfaces. Instead, passing through any given spacetime point, there is an infinite set of spatial hypersurfaces on which the deviations from homogeneity and isotropy are small, in the sense of the perturbation theory. [1] This freedom to choose the time slicing in the perturbed universe leads to gauge freedom in the perturbations which must be carefully dealt with. For example, merely stating that the density perturbation is such and such is meaningless because one is always free to choose spa-

[1]Note that there is an even larger set of hypersurfaces on which these deviations are not small due to the perverse choice of the hypersurface.

tial hypersurfaces on which the density perturbation is zero (of course, perturbations in other quantities are non-zero). Instead one must state that the density perturbation on such and such spatial hypersurfaces is such and such. We will see this more precisely below.

2.2.1 Gauge transformations



The perturbation in a scalar quantity $\tilde{\phi}$ at a point p in the perturbed universe $\tilde{U}$ is given by

$$\delta\phi\,(p) = \tilde{\phi}\,(p) - \phi\,(x^\mu) \tag{61}$$

where $\phi\,(x^\mu)$ is the value at the coordinate point x^μ in the spatially homogeneous and isotropic background universe U. For this to make sense we need to choose which coordinate point x^μ in the background universe U to associate with the point p in $\tilde{U}$. Different choices will give different $\phi\,(x^\mu)$ and hence different $\delta\phi\,(p)$. Given such a mapping $x^\mu(p) : \tilde{U} \to U$ we can unambiguously set

$$\delta\phi\,(p) = \tilde{\phi}\,(p) - \phi\,(x^\mu(p)) \tag{62}$$

However, there is no natural choice for this mapping.

The mapping $x^\mu(p) : \tilde{U} \to U$ induces a coordinate system $x^\mu(p)$ for $\tilde{U}$. A change in the mapping leads to a change in coordinates in $\tilde{U}$, and is called a gauge transformation to distinguish it from a genuine change in coordinates. Suppose the gauge transformation is given by

$$\hat{x}^\mu(p) = x^\mu(p) + \xi^\mu\,(x^\nu(p)) \tag{63}$$

Then

$$\hat{\delta\phi}\,(p) = \tilde{\phi}\,(p) - \phi\,(\hat{x}^\mu(p)) \tag{64}$$

$$= \delta\phi\,(p) - [\phi\,(\hat{x}^\mu(p)) - \phi\,(x^\mu(p))] \tag{65}$$

It is sufficient to consider infinitesimal gauge transformations, and so

$$\hat{\delta\phi}\,(p) = \delta\phi\,(p) - \xi^\nu \frac{\partial\phi}{\partial x^\nu}\,(x^\mu(p)) \tag{66}$$

The background universe U is spatially homogeneous and isotropic so this becomes

$$\hat{\delta\phi}\,(p) = \delta\phi\,(p) - \dot{\phi}\,(t(p))\;\xi^0\,(x^\mu(p)) \tag{67}$$

where we have taken $x^0 = t$. Another scalar quantity $\tilde{\psi}$ will transform similarly

$$\hat{\delta\psi}\,(p) = \delta\psi\,(p) - \dot{\psi}\,(t(p))\;\xi^0\,(x^\mu(p)) \tag{68}$$

There are two strategies one could take to deal with this gauge ambiguity. One could fix the gauge by choosing the perturbation in some physical quantity of interest to vanish. For example, one could choose the perturbation in $\tilde{\phi}$ to vanish. Temporarily treating ξ^0 as a finite gauge transformation, and working to first order in perturbation theory, Eq. (67) fixes

$$\xi^0 = \frac{\delta\phi}{\dot{\phi}} \tag{69}$$

but it is clear that ξ^0 is fixed to all orders in perturbation theory. This gauge fixing of ξ^0 fixes the constant time hypersurfaces in $\tilde{U}$ to be constant $\tilde{\phi}$ hypersurfaces with the time parameterization $t(p)$ fixed by $\phi\,(t(p)) = \tilde{\phi}\,(p)$.

Alternatively, one could just use gauge invariant quantities. For example

$$\Psi = \delta\psi - \frac{\dot{\psi}}{\dot{\phi}}\,\delta\phi \tag{70}$$

is gauge invariant. The physical interpretation of the gauge invariant quantity is clear: it is the perturbation in $\tilde{\psi}$ on constant $\tilde{\phi}$ hypersurfaces, but it can be evaluated on arbitrary hypersurfaces. Similarly $-\left(\dot{\phi}/\dot{\psi}\right)\Psi$ is the perturbation in $\tilde{\phi}$ on constant $\tilde{\psi}$ hypersurfaces.

Perturbations in tensor quantities can be handled in a similar way, except more care must be taken to evaluate the tensors at the same point in the same space. The mapping $x^\mu(p) : \tilde{U} \to U$ induces a mapping x^μ_* of tensors at the point p in $\tilde{U}$ to tensors at the coordinate point x^μ in U.

The perturbation in a tensor quantity $\tilde{T}$ at a point p in the perturbed universe $\tilde{U}$ is given by

$$\delta T\left(p\right) = \tilde{T}\left(p\right) - \left(x^\mu_*\right)^{-1} T\left(x^\mu(p)\right) \tag{71}$$

The gauge transformation

$$\hat{x}^\mu(p) = x^\mu(p) + \xi^\mu\left(x^\nu(p)\right) \tag{72}$$

gives

$$\hat{\delta T}\left(p\right) = \tilde{T}\left(p\right) - \left(\hat{x}^\mu_*\right)^{-1} T\left(\hat{x}^\mu(p)\right) \tag{73}$$

and so

$$\begin{aligned} &\hat{\delta T}\left(p\right) - \delta T\left(p\right) \\ &= -\left[\left(\hat{x}^\mu_*\right)^{-1} T\left(\hat{x}^\mu(p)\right) - \left(x^\mu_*\right)^{-1} T\left(x^\mu(p)\right)\right] \quad (74) \\ &= -\left(x^\mu_*\right)^{-1}\left[x^\mu_* \circ \left(\hat{x}^\mu_*\right)^{-1} T\left(\hat{x}^\mu(p)\right) - T\left(x^\mu(p)\right)\right] \quad (75) \end{aligned}$$

It is sufficient to consider infinitesimal gauge transformations, in which case

$$\hat{\delta T}\left(p\right) - \delta T\left(p\right) = -\left(x^\mu_*\right)^{-1} L_\xi T\left(x^\mu(p)\right) \tag{76}$$

where L is the Lie derivative. In terms of components, the Lie derivative of a tensor $T^\mu{}_\nu$ with respect to a vector ξ^σ is

$$\left(L_\xi T\right)^\mu{}_\nu = \frac{\partial T^\mu{}_\nu}{\partial x^\sigma}\xi^\sigma - T^\sigma{}_\nu \frac{\partial \xi^\mu}{\partial x^\sigma} + T^\mu{}_\sigma \frac{\partial \xi^\sigma}{\partial x^\nu} \tag{77}$$

and similarly for other tensors. The latter terms arise due to the rotation of $T\left(\hat{x}^\mu\right)$ as it is transported back to x^μ by $x^\mu_* \circ \left(\hat{x}^\mu_*\right)^{-1}$ along the field lines of the vector field $\xi^\mu\left(x^\nu\right)$.

2.2.2 Perturbation variables

We take a flat homogeneous and isotropic universe as our background

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j \tag{78}$$

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu - p(t)\left(g_{\mu\nu} - u_\mu u_\nu\right) \tag{79}$$

with $u^\mu = (1, 0, 0, 0)$. The perturbed universe is parameterized as

$$\tilde{ds}^2 = \left(1 + 2A\right) dt^2 - 2aB_i\, dt\, dx^i - a^2\left(\delta_{ij} + 2C_{ij}\right) dx^i dx^j \tag{80}$$

$$\tilde{T}_{\mu\nu} = \tilde{\rho}\tilde{u}_\mu\tilde{u}_\nu - \tilde{p}\left(\tilde{g}_{\mu\nu} - \tilde{u}_\mu\tilde{u}_\nu\right) + \tilde{\pi}_{\mu\nu} \tag{81}$$

where $\tilde{\rho}$ and $\tilde{p}$ are the energy density and pressure of the matter fluid in its rest frame, $\tilde{u}_\mu$ is its four-velocity and $\tilde{\pi}_{\mu\nu}$ is the anisotropic stress

$$\tilde{T}^\mu{}_\nu \tilde{u}^\nu = \tilde{\rho}\tilde{u}^\mu \tag{82}$$

$$\tilde{u}_\mu \tilde{u}^\mu = 1 \tag{83}$$

$$\tilde{\pi}_{\mu\nu}\tilde{u}^\nu = 0 = \tilde{\pi}^\mu_\mu \tag{84}$$

The matter perturbation variables are then

$$\delta\rho \equiv \tilde{\rho} - \rho \tag{85}$$

$$\delta p \equiv \tilde{p} - p \tag{86}$$

$$v^i \equiv a\tilde{u}^i \tag{87}$$

$$\pi_{ij} \equiv \frac{1}{a^2}\tilde{\pi}_{ij} \tag{88}$$

In the case of a scalar field we have

$$\phi = \phi(t) \tag{89}$$

and

$$\delta\phi \equiv \tilde{\phi} - \phi \tag{90}$$

From Eq. (76), perturbations in the metric transform as

$$\hat{\delta g}_{\mu\nu} - \delta g_{\mu\nu} = -g_{\mu\nu,\sigma}\xi^\sigma - g_{\sigma\nu}\xi^\sigma{}_{,\mu} - g_{\mu\sigma}\xi^\sigma{}_{,\nu} \tag{91}$$

Therefore

$$\hat{A} - A = -\dot{\xi}^0 \tag{92}$$

$$\hat{B}_i - B_i = \frac{1}{a}\xi^0{}_{,i} - a\dot{\xi}^i \tag{93}$$

$$\hat{C}_{ij} - C_{ij} = -H\delta_{ij}\xi^0 - \frac{1}{2}\left(\xi^i{}_{,j} + \xi^j{}_{,i}\right) \tag{94}$$

Similarly

$$\hat{\delta\rho} - \delta\rho = -\dot{\rho}\xi^0 \tag{95}$$

$$\hat{\delta p} - \delta p = -\dot{p}\xi^0 \tag{96}$$

$$\hat{v}^i - v^i = a\dot{\xi}^i \tag{97}$$

$$\hat{\pi}_{ij} - \pi_{ij} = 0 \tag{98}$$

and

$$\hat{\delta\phi} - \delta\phi = -\dot{\phi}\xi^0 \tag{99}$$

Note that π_{ij} is gauge invariant. This is because $\tilde{\pi}_{\mu\nu}$ is zero in the background.

2.2.3 Scalar, vector and tensor modes

We can decompose the perturbation variables into scalar, vector and tensor variables as

$$A = A \tag{100}$$

$$B_i = \frac{1}{a}\partial_i B + B_i^{(\mathrm{v})}, \qquad \partial_i B_i^{(\mathrm{v})} = 0 \tag{101}$$

$$C_{ij} = \mathcal{R}\delta_{ij} + \frac{1}{a^2}\partial_i\partial_j C + \frac{1}{2a}\left(\partial_i C_j + \partial_j C_i\right) + C_{ij}^{(\mathrm{t})} \tag{102}$$

$$\partial_i C_i = 0, \qquad \partial_i C_{ij}^{(\mathrm{t})} = 0, \qquad C_{ii}^{(\mathrm{t})} = 0 \tag{103}$$

$$\delta\rho = \delta\rho \tag{104}$$

$$\delta p = \delta p \tag{105}$$

$$v^i = \frac{1}{a}\partial_i v + v^i_{(\mathrm{v})}, \qquad \partial_i v^i_{(\mathrm{v})} = 0 \tag{106}$$

$$\pi_{ij} = \varpi\delta_{ij} + \frac{1}{a^2}\partial_i\partial_j\pi + \frac{1}{2a}\left(\partial_i\pi_j + \partial_j\pi_i\right) + \pi_{ij}^{(\mathrm{t})} \tag{107}$$

$$\partial_i\pi_i = 0, \qquad \partial_i\pi_{ij}^{(\mathrm{t})} = 0, \qquad \pi_{ii}^{(\mathrm{t})} = 0 \tag{108}$$

where ϖ can be eliminated using Eq. (84), and

$$\delta\phi = \delta\phi \tag{109}$$

To first order in perturbation theory, the equations of motion for the scalar, vector and tensor perturbations decouple.

Decomposing ξ^μ as

$$\xi^0 = T \tag{110}$$

$$\xi^i = \frac{1}{a}\partial_i L + L^i\,, \qquad \partial_i L^i = 0 \tag{111}$$

Eqs. (92) to (99) give

$$\hat{A} - A = -\dot{T} \tag{112}$$
$$\hat{B} - B = T - a\dot{L} + \dot{a}L \tag{113}$$
$$\hat{\mathcal{R}} - \mathcal{R} = -HT \tag{114}$$
$$\hat{C} - C = -aL \tag{115}$$
$$\hat{\delta\rho} - \delta\rho = -\dot{\rho}T \tag{116}$$
$$\hat{\delta p} - \delta p = -\dot{p}T \tag{117}$$
$$\hat{v} - v = a\dot{L} - \dot{a}L \tag{118}$$
$$\hat{\pi} - \pi = 0 \tag{119}$$
$$\hat{\delta\phi} - \delta\phi = -\dot{\phi}T \tag{120}$$
$$\hat{B}_i^{(\mathrm{v})} - B_i^{(\mathrm{v})} = -a\dot{L}^i \tag{121}$$
$$\hat{C}_i - C_i = -aL^i \tag{122}$$
$$\hat{v}^i_{(\mathrm{v})} - v^i_{(\mathrm{v})} = a\dot{L}^i \tag{123}$$
$$\hat{\pi}_i - \pi_i = 0 \tag{124}$$
$$\hat{C}_{ij}^{(\mathrm{t})} - C_{ij}^{(\mathrm{t})} = 0 \tag{125}$$
$$\hat{\pi}_{ij}^{(\mathrm{t})} - \pi_{ij}^{(\mathrm{t})} = 0 \tag{126}$$

2.2.4 Equations of motion

The Einstein equation gives

$$3H\left(\dot{\mathcal{R}} - HA\right) + \left(\frac{k}{a}\right)^2\left[\mathcal{R} - H\left(\dot{C} - 2HC - B\right)\right] = \frac{1}{2}\,\delta\rho \tag{127}$$

$$\dot{\mathcal{R}} - HA = -\dot{H}\,(v + B) \tag{128}$$

$$\left(\dot{C} - 2HC - B\right)^{\cdot} + H\left(\dot{C} - 2HC - B\right) - \mathcal{R} - A = \pi \tag{129}$$

$$\left(\dot{\mathcal{R}} - HA\right)^{\cdot} + 3H\left(\dot{\mathcal{R}} - HA\right) - \dot{H}A = -\frac{1}{2}\,\delta p + \frac{1}{3}\left(\frac{k}{a}\right)^2\pi \tag{130}$$

$$\left(\frac{k}{a}\right)^2\left(\dot{C}_i - HC_i - B_i^{(\mathrm{v})}\right) = -4\dot{H}\left(v^i_{(\mathrm{v})} + B_i^{(\mathrm{v})}\right) \tag{131}$$

$$\left(\dot{C}_i - HC_i - B_i^{(\mathrm{v})}\right)^{\cdot} + 2H\left(\dot{C}_i - HC_i - B_i^{(\mathrm{v})}\right) = \pi_i \tag{132}$$

$$\ddot{C}_{ij}^{(\mathrm{t})} + 3H\dot{C}_{ij}^{(\mathrm{t})} + \left(\frac{k}{a}\right)^2 C_{ij}^{(\mathrm{t})} = \pi_{ij}^{(\mathrm{t})} \tag{133}$$

References

1. J. M. Bardeen, Physical Review D22 (1980) 1882-1905.
2. H. Kodama and M. Sasaki, Progress of Theoretical Physics Supplement 78 (1984) 1-166.