2 General Relativity

2.1 Homogeneous and isotropic expanding universe

2.1.1 The metric

To measure distances in spacetime we need a metric $g_{\mu\nu}$. The infinitesimal spacetime line element is then

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{21}$$

where x^μ, $\mu = 0, 1, 2, 3$, are the spacetime coordinates.

For a spatially homogeneous and isotropic universe there is a natural space time splitting given by choosing the time to be constant on the homogeneous and isotropic spatial hypersurfaces.

Note that this will cease to be the case when we introduce perturbations. Note also that the maximally symmetric spacetimes, de Sitter space, anti-de Sitter space and Minkowski space, do not have unique choices for the homogeneous and isotropic spatial hypersurfaces, and so do not have a unique natural space time splitting. Both these points will be important later on.

Let t be the proper time defined by the homogeneous and isotropic spatial hypersurfaces. Then the metric can be written in the form

$$ds^2 = dt^2 - a(t)^2 h_{ij} dx^i dx^j \tag{22}$$

where $a(t)$ is the scale factor of the universe (we will assume $a > 0$, and $\dot{a} > 0$ corresponding to an expanding universe); h_{ij} is a constant metric on the homogeneous and isotropic, but in general curved, spatial hypersurfaces; x^i, $i = 1, 2, 3$, are comoving spatial coordinates (they are called 'comoving' because they expand with the universe).

The constant comoving intrinsic curvature of the spatial hypersurfaces can be positive, negative or zero, and can be scaled to be 1, −1 or 0. The spatial metric h_{ij} can then be written in the form

$$h_{ij} dx^i dx^j = dr^2 + f^2(r) \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \tag{23}$$

where

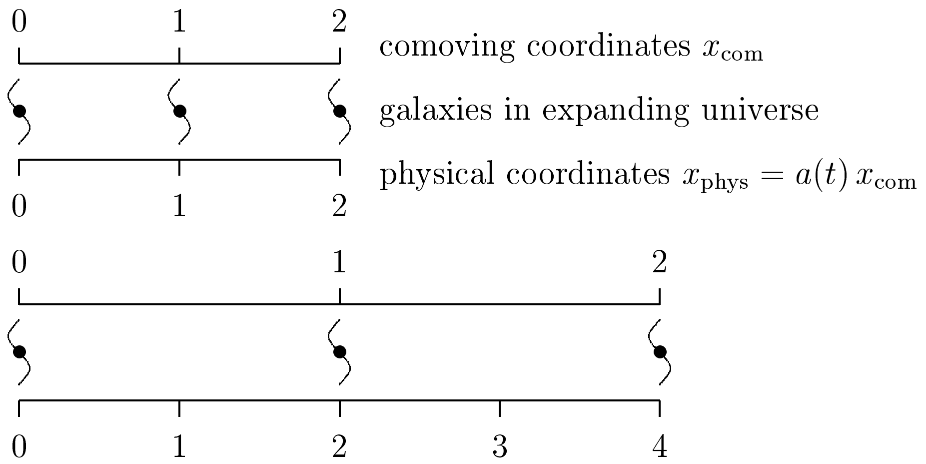
$$f(r) = \begin{cases} \sin r & \text{if} \quad K_{\text{com}} = 1 \\ \sinh r & \text{if} \quad K_{\text{com}} = -1 \\ r & \text{if} \quad K_{\text{com}} = 0 \end{cases} \tag{24}$$

The physical intrinsic curvature of the spatial hypersurfaces K is related to the constant comoving intrinsic curvature K_{com} by

$$K = \frac{K_{\text{com}}}{a(t)^2} \tag{25}$$

If $K = 0$ one can also write the metric in the form

$$ds^2 = dt^2 - a(t)^2 \left(dx^2 + dy^2 + dz^2 \right) \tag{26}$$



A physical spatial distance x_{phys} is related to a comoving distance x_{com} by

$$x_{\mathrm{phys}} = a(t)\, x_{\mathrm{com}} \tag{27}$$

For constant comoving distances

$$\dot{x}_{\mathrm{phys}} = \dot{a} x_{\mathrm{com}} = \frac{\dot{a}}{a} x_{\mathrm{phys}} \tag{28}$$

$$= H x_{\mathrm{phys}} \qquad \text{Hubble's Law} \tag{29}$$

where

$$H \equiv \frac{\dot{a}}{a} \tag{30}$$

is the Hubble parameter.

The current value of the Hubble parameter is often parametrized as

$$H_0 \equiv 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 2.133h \times 10^{-42}\,\mathrm{GeV} \tag{31}$$

Observations give $h \simeq 0.65$ with an error of about 10%.

For

$$x_{\mathrm{phys}} > \frac{1}{H} = \text{Hubble distance} \tag{32}$$

we have

$$\dot{x}_{\mathrm{phys}} > 1 \tag{33}$$

so that things more than a Hubble distance away are out of physical contact and are said to be 'beyond the horizon'. Also

$$\frac{d}{dt}\left(\frac{x_{\mathrm{phys}}}{1/H}\right) = \frac{d}{dt}\left(aHx_{\mathrm{com}}\right) = \ddot{a} x_{\mathrm{com}} \tag{34}$$

for $\dot{x}_{\mathrm{com}} = 0$. Therefore, if $\ddot{a} < 0$ comoving scales move inside the horizon, and if $\ddot{a} > 0$ they move outside the horizon.

2.1.2 The Einstein equation

The dynamics of the universe is governed by the Einstein equation

$$G_{\mu\nu} = T_{\mu\nu} \tag{35}$$

For a spatially homogeneous and isotropic universe

$$T_\mu{}^\nu = \mathrm{diag}\,(\rho, -p, -p, -p) \tag{36}$$

where ρ is the energy density and p is the pressure (we will assume that $\rho > 0$). The Einstein equation then gives

$$3H^2 + 3K = \rho \tag{37}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\,(\rho + 3p) \tag{38}$$

or

$$\dot{H} = -\frac{1}{2}\,(\rho + p) + K \tag{39}$$

The critical density ρ_c is defined as the energy density of a flat universe

$$\rho_\mathrm{c} \equiv 3H^2 \tag{40}$$

The current value of ρ_c is

$$\rho_\mathrm{c} = \left(3.000 h^{1/2} \times 10^{-3}\,\mathrm{eV}\right)^4 = 2.301 h^2 \times 10^{-120} \simeq 4\,\mathrm{GeV\,m^{-3}} \tag{41}$$

It is conventional to measure densities relative to the critical density

$$\Omega_X \equiv \frac{\rho_X}{\rho_\mathrm{c}} \tag{42}$$

2.1.3 Matter, fields, ...

Differentiating Eq. (37) and using Eq. (39) gives

$$\frac{d\ln\rho}{d\ln a} = -3\left(1 + \frac{p}{\rho}\right) \tag{43}$$

which is just a rewriting of $dE = -p\,dV$ and can be derived from $T^{\mu\nu}{}_{;\nu} = 0$. If

$$w \equiv \frac{p}{\rho} = \mathrm{constant} \tag{44}$$

then Eq. (43) gives

$$\rho \propto a^{-3(1+w)} \tag{45}$$

If the universe is dominated by such material, then the Einstein equation gives

$$H = \frac{2}{3(1+w)t} \tag{46}$$

and

$$a \propto t^{\frac{2}{3(1+w)}} \tag{47}$$

Now $w = \frac{1}{3}, 0, -1$ for radiation, matter, vacuum energy, respectively, so

$$\rho_\mathrm{rad} \propto a^{-4}\,,\ \rho_\mathrm{mat} \propto a^{-3}\,,\ K \propto a^{-2}\,,\ \rho_\mathrm{vac} = \mathrm{constant} \tag{48}$$

and for a radiation, matter, (negative) curvature, vacuum energy, respectively, dominated universe

$$H_\mathrm{rad} = \frac{1}{2t}\,,\ H_\mathrm{mat} = \frac{2}{3t}\,,\ H_K = \frac{1}{t}\,,\ H_\mathrm{vac} = \mathrm{constant} \tag{49}$$

and

$$a_\mathrm{rad} \propto t^{1/2}\,,\ a_\mathrm{mat} \propto t^{2/3}\,,\ a_K \propto t\,,\ a_\mathrm{vac} \propto e^{Ht} \tag{50}$$

The energy-momentum tensor for vacuum energy is

$$T_{\mu\nu} = \Lambda g_{\mu\nu} \tag{51}$$

and so could be put on the left hand side of the Einstein equation as a 'cosmological constant'. We will keep it on the right hand side though.

The action for a real scalar field $\phi(x^\mu)$ is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] \tag{52}$$

where g is the determinant of the metric and $g^{\mu\nu}$ is the inverse metric $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$. The energy-momentum tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi - V(\phi) \right] \tag{53}$$

Therefore, in a homogeneous and isotropic universe

$$\rho_\phi = T^0_0 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \frac{1}{a^2} \left\langle h^{ij} \partial_i \phi \, \partial_j \phi \right\rangle + V \tag{54}$$

$$p_\phi = -\frac{1}{3} T^i_i = \frac{1}{2} \dot{\phi}^2 - \frac{1}{6} \frac{1}{a^2} \left\langle h^{ij} \partial_i \phi \, \partial_j \phi \right\rangle - V \tag{55}$$

Note that any vacuum energy can be absorbed into the scalar field's potential energy $V(\phi)$.

The action for a massless vector field $A_\mu(x^\nu)$ is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right] \tag{56}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Its energy-momentum tensor is

$$T_{\mu\nu} = -g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{1}{4} g_{\mu\nu} g^{\rho\sigma} g^{\eta\kappa} F_{\rho\eta} F_{\sigma\kappa} \tag{57}$$

Therefore, in a homogeneous and isotropic universe

$$\rho_\gamma = \frac{1}{2} \frac{1}{a^2} \left\langle h^{ij} F_{0i} F_{0j} \right\rangle + \frac{1}{4} \frac{1}{a^4} \left\langle h^{ij} h^{kl} F_{ik} F_{jl} \right\rangle \tag{58}$$

$$p_\gamma = \frac{1}{3} \rho_\gamma \tag{59}$$

The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = \rho u_\mu u_\nu - p \left(g_{\mu\nu} - u_\mu u_\nu \right) \tag{60}$$

where $u^\mu = g^{\mu\nu} u_\nu$ is the fluid's four velocity, $u_\mu u^\mu = 1$.