Final Exam - 2pm Saturday 10th June, Creative 302

Your answers should be clear and concise. They should start from basic principles and proceed logically.

Q1. Show that the curvature tensor identities

$$R_{\mathbf{abcd}} = -R_{\mathbf{bacd}} \tag{Q1.1}$$
$$R_{\mathbf{abcd}} = -R_{\mathbf{abdc}} \tag{Q1.2}$$
$$R_{[\mathbf{abc}]\mathbf{d}} = 0 \tag{Q1.3}$$

are equivalent to

$$R_{\mathbf{abcd}} = -R_{\mathbf{bacd}} \tag{Q1.4}$$
$$R_{\mathbf{abcd}} = R_{\mathbf{cdab}} \tag{Q1.5}$$
$$R_{[\mathbf{abcd}]} = 0 \tag{Q1.6}$$

A1. Eq. (Q1.1) identically implies Eq. (Q1.4), Homework Q1.2 showed that Eqs. (Q1.1), (Q1.2) and (Q1.3) imply Eq. (Q1.5), and Eq. (Q1.3) trivially implies Eq. (Q1.6), thus Eqs. (Q1.1), (Q1.2) and (Q1.3) imply Eqs. (Q1.4), (Q1.5) and (Q1.6).

Eq. (Q1.4) identically implies Eq. (Q1.1). Using Eqs. (Q1.4) and (Q1.5),

$$\begin{aligned} R_{\mathbf{abcd}} &= R_{\mathbf{cdab}} && \text{(A1.1)} \\ &= -R_{\mathbf{dcab}} && \text{(A1.2)} \\ &= -R_{\mathbf{abdc}} && \text{(A1.3)} \end{aligned}$$

therefore Eqs. (Q1.4) and (Q1.5) imply Eq. (Q1.2). Using Eqs. (Q1.6), (Q1.5) and (A1.3) respectively,

$$\begin{aligned} 0 &= 4R_{[\mathbf{abcd}]} && \text{(A1.4)} \\ &= R_{[\mathbf{abc}]\mathbf{d}} - R_{[\mathbf{ab}|\mathbf{d}|\mathbf{c}]} + R_{[\mathbf{a}|\mathbf{d}|\mathbf{bc}]} - R_{\mathbf{d}[\mathbf{abc}]} && \text{(A1.5)} \\ &= R_{[\mathbf{abc}]\mathbf{d}} - R_{[\mathbf{ab}|\mathbf{d}|\mathbf{c}]} + R_{[\mathbf{bca}]\mathbf{d}} - R_{[\mathbf{bc}|\mathbf{d}|\mathbf{a}]} && \text{(A1.6)} \\ &= R_{[\mathbf{abc}]\mathbf{d}} + R_{[\mathbf{abc}]\mathbf{d}} + R_{[\mathbf{bca}]\mathbf{d}} + R_{[\mathbf{bca}]\mathbf{d}} && \text{(A1.7)} \\ &= 4R_{[\mathbf{abc}]\mathbf{d}} && \text{(A1.8)} \end{aligned}$$

where $|\mathbf{d}|$ indicates that $\mathbf{d}$ is excluded from the antisymmetrization. Therefore Eqs. (Q1.4), (Q1.5) and (Q1.6) imply Eq. (Q1.3). Thus Eqs. (Q1.4), (Q1.5) and (Q1.6) imply Eqs. (Q1.1), (Q1.2) and (Q1.3).

Therefore Eqs. (Q1.1), (Q1.2) and (Q1.3) are equivalent to Eqs. (Q1.4), (Q1.5) and (Q1.6).

Q2. Determine the nature of the spacetime with metric

$$d\tau^2 = dt^2 - t^2\left(d\xi^2 + \sinh^2\xi\, d\Omega^2\right) \tag{Q2.1}$$

A2. The spacetime appears to be an expanding universe with scale factor $a(t) = t$ and negatively curved spatial hypersurfaces. However, the metric of Eq. (Q2.1) can be obtained as the $H \to 0$ limit of the de Sitter metric Eq. (2.5.5)

$$d\tau^2 = dt^2 - H^{-2}\sinh^2(Ht)\left(d\xi^2 + \sinh^2\xi\, d\Omega^2\right) \tag{A2.1}$$

and so corresponds to the $H \to 0$ limit of de Sitter space, i.e. Minkowski space. For example, the $H \to 0$ limit of the de Sitter metric Eq. (2.5.4)

$$d\tau^2 = dt'^2 - \exp(2Ht')\left(dr^2 + r^2\, d\Omega^2\right) \tag{A2.2}$$

or Eq. (2.5.6)

$$d\tau^2 = \left(1 - H^2r^2\right)dt'^2 - \left(\frac{dr^2}{1 - H^2r^2} + r^2\, d\Omega^2\right) \tag{A2.3}$$

is the Minkowski metric

$$d\tau^2 = dt'^2 - dr^2 - r^2\, d\Omega^2 \tag{A2.4}$$

More directly, the change in coordinates

$$t' = t\cosh\xi \tag{A2.5}$$
$$r = t\sinh\xi \tag{A2.6}$$

transforms the Minkowski metric Eq. (A2.4) into the metric of Eq. (Q2.1) and so both describe the same spacetime, Minkowski space.

Q3. Consider the spacetime with metric

$$d\tau^2 = \left(1 - \frac{2GM}{r} - H^2r^2\right)dt^2 - \left(1 - \frac{2GM}{r} - H^2r^2\right)^{-1}dr^2 - r^2\, d\Omega^2 \tag{Q3.1}$$

(a) Describe the properties of the spacetime.

(b) Draw its Penrose diagram.

(c) Describe the possible circular orbits of a particle moving freely in this spacetime.

A3. (a) The spacetime is static and spherically symmetric, and has a physical singularity at $r = 0$.

g_{tt} is zero and g_{rr} diverges when

$$g_{tt} = 1 - \frac{2GM}{r} - H^2r^2 = 0 \tag{A3.1}$$

which has two solutions if

$$GMH < \frac{1}{3\sqrt{3}} \tag{A3.2}$$

corresponding to an event horizon at $r = r_\mathrm{e}$ and a cosmological horizon at $r = r_\mathrm{c}$, with

$$2GM < r_\mathrm{e} < 3GM < \frac{1}{\sqrt{3}\,H} < r_\mathrm{c} < \frac{1}{H} \tag{A3.3}$$

The case $3\sqrt{3}\,GMH \geq 1$ is unphysical corresponding to a naked singularity.

(b) The Penrose diagram is a combination of the Schwarzschild and de Sitter Penrose diagrams, see Figure A3.1.

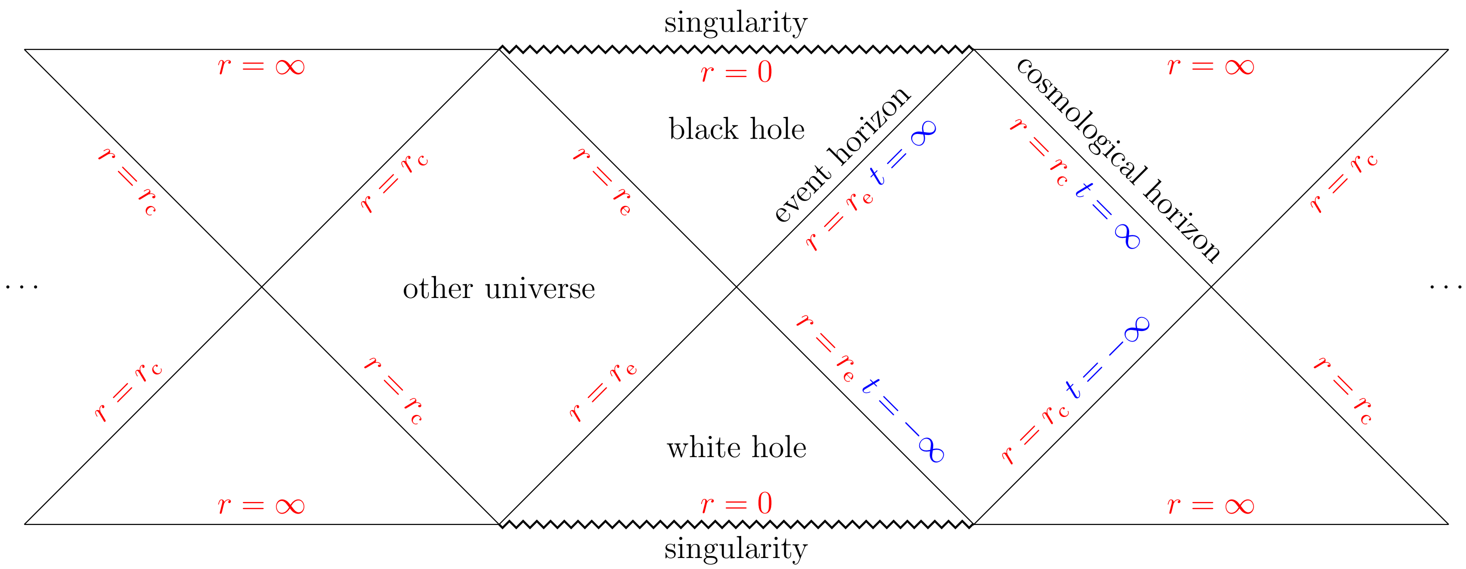

Figure A3.1: Penrose diagram for Schwarzschild de Sitter spacetime. The pattern repeats indefinitely.

(c) Following Section 2.6.1, fixing $\theta = \pi/2$ and using the results of Homework 4, time translational and rotational symmetries give the conserved quantities

$$E = e_t^{\mathbf{a}} p_{\mathbf{a}} = m g_{t\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m\left(1 - \frac{2GM}{r} - H^2 r^2\right)\frac{dt}{d\tau} \tag{A3.4}$$

and

$$L = -e_\phi^{\mathbf{a}} p_{\mathbf{a}} = -m g_{\phi\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m r^2 \frac{d\phi}{d\tau} \tag{A3.5}$$

while the radial motion can be determined using

$$m^2 = g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}} = g^{tt} p_t^2 + g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \tag{A3.6}$$

$$= \left(1 - \frac{2GM}{r} - H^2 r^2\right)^{-1}\left[E^2 - m^2\left(\frac{dr}{d\tau}\right)^2\right] - \frac{L^2}{r^2} \tag{A3.7}$$

therefore

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{E^2 - m^2}{2m} \tag{A3.8}$$

where the effective potential

$$V(r) = -\frac{GML^2}{mr^3} + \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{H^2L^2}{2m} - \frac{1}{2}mH^2r^2 \tag{A3.9}$$

and

$$V'(r) = \frac{3GML^2}{mr^4} - \frac{L^2}{mr^3} + \frac{GMm}{r^2} - mH^2r \tag{A3.10}$$

In general, the effective potential will have two maxima and a minimum, the first maximum at

$$r \sim 3GM\left[1 + 3\left(\frac{GMm}{L}\right)^2 + \dots\right] \tag{A3.11}$$

corresponding to the Schwarzschild unstable circular orbit, the minimum at

$$r \sim \frac{L^2}{GMm^2} \tag{A3.12}$$

to the Newtonian stable circular orbit, and the second maximum at

$$r \sim \left(\frac{GM}{H^2}\right)^{\frac{1}{3}} - \frac{L^2}{3GMm^2} + \dots \tag{A3.13}$$

to a de Sitter unstable circular orbit balancing the Newtonian attraction against the de Sitter repulsion. If

$$L \lesssim 2\sqrt{3}\,GMm \tag{A3.14}$$

then the Schwarzschild and Newtonian orbits disappear leaving the de Sitter unstable circular orbit. If

$$L \gtrsim \frac{\sqrt{3}\,m}{2}\left(\frac{G^2M^2}{2H}\right)^{\frac{1}{3}} \tag{A3.15}$$

then the de Sitter and Newtonian orbits disappear leaving the Schwarzschild unstable circular orbit.

Q4. Show that a potential of the form

$$V(\phi) = V_0 - \frac{1}{2}m^2\phi^2 + \dots \tag{Q4.1}$$

gives rise to eternal inflation if $m^2 < 6V_0$ in units where $\hbar = c = 8\pi G = 1$.

A4. Eqs. (3.2.168), (3.2.176) and (3.2.185) give

$$\phi_k \sim \frac{1}{a}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu} \tag{A4.1}$$

where, for the case of Eq. (Q4.1), Eq. (3.2.183) gives

$$\nu = \sqrt{\frac{9}{4} + \frac{m^2}{H^2}} \tag{A4.2}$$

while the spatial volume increases as

$$V \sim a^3 \tag{A4.3}$$

Therefore the field distribution spreads more slowly than the volume increases, and hence the inflating volume increases and one gets eternal inflation, if

$$\nu - \frac{3}{2} < 3 \tag{A4.4}$$

or using Eq. (A4.2)

$$\frac{m^2}{H^2} < 18 \tag{A4.5}$$

or using Eq. (3.1.8) with $\rho \simeq V_0$

$$\frac{m^2}{V_0} < 6 \tag{A4.6}$$