Homework 1 - Curvature tensor

Q1.1. Calculate

$$(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v^\mathbf{c} \tag{Q1.1.1}$$

in two different ways, and hence show that

$$R_\mathbf{abcd} = -R_\mathbf{abdc} \tag{Q1.1.2}$$

Find all possible distinct contractions of the metric with the curvature tensor.

A1.1. Using Eqs. (1.2.5) and (1.2.7),

$$(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v^\mathbf{c} = g^\mathbf{ce}\,(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v_\mathbf{e} = g^\mathbf{ce} R_\mathbf{abe}{}^\mathbf{d} v_\mathbf{d} = g^\mathbf{ce} R_\mathbf{abed} v^\mathbf{d} \tag{A1.1.1}$$

and, using Eqs. (1.2.6) and (1.2.7),

$$0 = (\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\left(v^\mathbf{d}\omega_\mathbf{d}\right) \tag{A1.1.2}$$

$$= \omega_\mathbf{d}\,(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v^\mathbf{d} + v^\mathbf{d}\,(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\,\omega_\mathbf{d} \tag{A1.1.3}$$

$$= \omega_\mathbf{c}\,(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v^\mathbf{c} + v^\mathbf{d} R_\mathbf{abd}{}^\mathbf{c}\omega_\mathbf{c} \tag{A1.1.4}$$

therefore

$$(\nabla_\mathbf{a}\nabla_\mathbf{b} - \nabla_\mathbf{b}\nabla_\mathbf{a})\, v^\mathbf{c} = -R_\mathbf{abd}{}^\mathbf{c} v^\mathbf{d} = -g^\mathbf{ce} R_\mathbf{abde} v^\mathbf{d} \tag{A1.1.5}$$

Comparing Eqs. (A1.1.1) and (A1.1.5) gives Eq. (Q1.1.2).

Eq. (1.2.7) implies

$$R_\mathbf{abcd} = -R_\mathbf{bacd} \tag{A1.1.6}$$

and Eqs. (A1.1.6) and (Q1.1.2) imply

$$g^\mathbf{cd} R_\mathbf{cdab} = g^\mathbf{cd} R_\mathbf{abcd} = 0 \tag{A1.1.7}$$

and

$$g^\mathbf{cd} R_\mathbf{acbd} = -g^\mathbf{cd} R_\mathbf{acdb} = -g^\mathbf{cd} R_\mathbf{cabd} = g^\mathbf{cd} R_\mathbf{cadb} \equiv R_\mathbf{ab} \tag{A1.1.8}$$

which is called the **Ricci tensor**. Further

$$g^\mathbf{ab} R_\mathbf{ab} \equiv R \tag{A1.1.9}$$

which is called the **Ricci scalar**.

Q1.2. Use the curvature tensor identity

$$R_\mathbf{abc}{}^\mathbf{d} + R_\mathbf{bca}{}^\mathbf{d} + R_\mathbf{cab}{}^\mathbf{d} = 0 \tag{Q1.2.1}$$

to show that

$$R_\mathbf{abcd} = R_\mathbf{cdab} \tag{Q1.2.2}$$

A1.2. Using Eqs. (Q1.2.1), (Q1.1.2) and (A1.1.6),

$$
\begin{aligned}
0 &= g_{\mathbf{de}} \left(R_{\mathbf{abc}}{}^{\mathbf{e}} + R_{\mathbf{bca}}{}^{\mathbf{e}} + R_{\mathbf{cab}}{}^{\mathbf{e}} \right) && \text{(A1.2.1)} \\
&= R_{\mathbf{abcd}} + R_{\mathbf{bcad}} + R_{\mathbf{cabd}} && \text{(A1.2.2)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{bcda}} - R_{\mathbf{cadb}} && \text{(A1.2.3)} \\
&= R_{\mathbf{abcd}} + R_{\mathbf{cdba}} + R_{\mathbf{dbca}} + R_{\mathbf{adcb}} + R_{\mathbf{dcab}} && \text{(A1.2.4)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{cdab}} - R_{\mathbf{dbac}} - R_{\mathbf{adbc}} - R_{\mathbf{cdab}} && \text{(A1.2.5)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{cdab}} + R_{\mathbf{badc}} - R_{\mathbf{cdab}} && \text{(A1.2.6)} \\
&= 2 \left(R_{\mathbf{abcd}} - R_{\mathbf{cdab}} \right) && \text{(A1.2.7)}
\end{aligned}
$$