Homework 2 - Bases and coordinates

Q2.1. Show that

(a)
$$v^\alpha = e^\alpha_{\mathbf{a}} v^{\mathbf{a}} \tag{Q2.1.1}$$

(b)
$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_\alpha v^\alpha \tag{Q2.1.2}$$

(c)
$$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha = \delta^{\mathbf{b}}_{\mathbf{a}} \tag{Q2.1.3}$$

and explain the meaning of all the terms.

A2.1. (a) Using Eqs. (1.2.11) and (1.2.12),

$$e^\alpha_{\mathbf{a}} v^{\mathbf{a}} = e^\alpha_{\mathbf{a}} v^\beta e^{\mathbf{a}}_\beta = v^\beta \delta^\alpha_\beta = v^\alpha \tag{A2.1.1}$$

v^α is the α component of the vector $v^{\mathbf{a}}$ and $e^\alpha_{\mathbf{a}} v^{\mathbf{a}}$ is the α basis covector contracted with the vector $v^{\mathbf{a}}$.

(b) Using Eqs. (1.2.14), (1.2.11) and (1.2.12),

$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_\alpha e^\alpha_{\mathbf{a}} v^\beta e^{\mathbf{a}}_\beta = \omega_\alpha v^\beta \delta^\alpha_\beta = \omega_\alpha v^\alpha \tag{A2.1.2}$$

$\omega_{\mathbf{a}} v^{\mathbf{a}}$ is the covector $\omega_{\mathbf{a}}$ contracted with the vector $v^{\mathbf{a}}$ and $\omega_\alpha v^\alpha$ is the sum of the products of the components of $\omega_{\mathbf{a}}$ and $v^{\mathbf{a}}$.

(c) Using Eqs. (Q2.1.1) and (1.2.11),

$$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha v^{\mathbf{a}} = e^{\mathbf{b}}_\alpha v^\alpha = v^{\mathbf{b}} \tag{A2.1.3}$$

therefore
$$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha = \delta^{\mathbf{b}}_{\mathbf{a}} \tag{A2.1.4}$$

$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha$ is the sum of the products of the basis covectors and vectors and $\delta^{\mathbf{b}}_{\mathbf{a}}$ is the identity tensor.

Q2.2. Let $e^{\mathbf{a}}_r$ and $e^{\mathbf{a}}_\theta$ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $e^{\mathbf{a}}_{\hat{r}}$ and $e^{\mathbf{a}}_{\hat{\theta}}$ be the orthonormal basis vectors proportional to $e^{\mathbf{a}}_r$ and $e^{\mathbf{a}}_\theta$.

(a) Express the metric $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases.

(b) Express the coordinate basis vectors and covectors in terms of the orthonormal basis vectors and covectors.

(c) Draw simple diagrams illustrating $e^{\mathbf{a}}_r$, $e^{\mathbf{a}}_\theta$, $e^r_{\mathbf{a}}$, $e^\theta_{\mathbf{a}}$ $e^{\mathbf{a}}_{\hat{r}}$, $e^{\mathbf{a}}_{\hat{\theta}}$, $e^{\hat{r}}_{\mathbf{a}}$ and $e^{\hat{\theta}}_{\mathbf{a}}$.

A2.2. (a) In polar coordinates

$$g_{\alpha\beta}dx^\alpha dx^\beta = ds^2 = dr^2 + r^2 d\theta^2 \tag{A2.2.1}$$

therefore $g_{rr} = 1$, $g_{r\theta} = 0$ and $g_{\theta\theta} = r^2$, and using $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$ gives $g^{rr} = 1$, $g^{r\theta} = 0$ and $g^{\theta\theta} = r^{-2}$. Therefore

$$g_{\mathbf{ab}} = g_{\alpha\beta} e^\alpha_{\mathbf{a}} e^\beta_{\mathbf{b}} = e^r_{\mathbf{a}} e^r_{\mathbf{b}} + r^2 e^\theta_{\mathbf{a}} e^\theta_{\mathbf{b}} \tag{A2.2.2}$$

and

$$g^{\mathbf{ab}} = g^{\alpha\beta} e^{\mathbf{a}}_\alpha e^{\mathbf{b}}_\beta = e^{\mathbf{a}}_r e^{\mathbf{b}}_r + \frac{1}{r^2} e^{\mathbf{a}}_\theta e^{\mathbf{b}}_\theta \tag{A2.2.3}$$

In an orthonormal basis

$$g_{\alpha\beta} = g_{\mathbf{ab}} e^{\mathbf{a}}_\alpha e^{\mathbf{b}}_\beta = \delta_{\alpha\beta} \tag{A2.2.4}$$

therefore

$$g_{\mathbf{ab}} = e^{\hat{r}}_{\mathbf{a}} e^{\hat{r}}_{\mathbf{b}} + e^{\hat{\theta}}_{\mathbf{a}} e^{\hat{\theta}}_{\mathbf{b}} \tag{A2.2.5}$$

and

$$g^{\mathbf{ab}} = e^{\mathbf{a}}_{\hat{r}} e^{\mathbf{b}}_{\hat{r}} + e^{\mathbf{a}}_{\hat{\theta}} e^{\mathbf{b}}_{\hat{\theta}} \tag{A2.2.6}$$

(b) Comparing Eqs. (A2.2.2) and (A2.2.5) and Eqs. (A2.2.3) and (A2.2.6) gives

$$e^r_{\mathbf{a}} = e^{\hat{r}}_{\mathbf{a}} \quad , \quad e^\theta_{\mathbf{a}} = \frac{1}{r} e^{\hat{\theta}}_{\mathbf{a}} \tag{A2.2.7}$$

$$e^{\mathbf{a}}_r = e^{\mathbf{a}}_{\hat{r}} \quad , \quad e^{\mathbf{a}}_\theta = r e^{\mathbf{a}}_{\hat{\theta}} \tag{A2.2.8}$$

(c) See Figure A2.2.1.

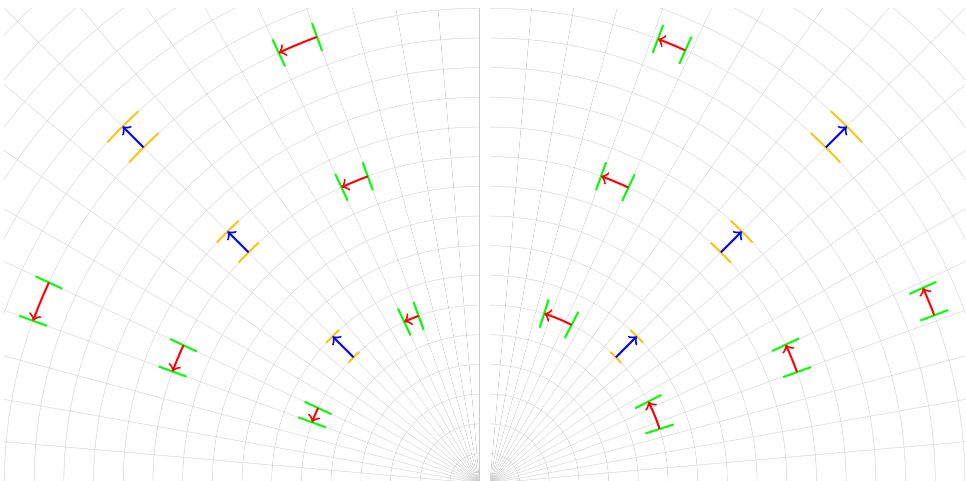

Figure A2.2.1: Left: polar coordinate basis, $e^{\mathbf{a}}_r$, $e^{\mathbf{a}}_\theta$, $e^r_{\mathbf{a}}$, $e^\theta_{\mathbf{a}}$. Right: polar orthonormal basis, $e^{\mathbf{a}}_{\hat{r}}$, $e^{\mathbf{a}}_{\hat{\theta}}$, $e^{\hat{r}}_{\mathbf{a}}$, $e^{\hat{\theta}}_{\mathbf{a}}$.