Homework 4 - Lie derivative

Q4.1. Show that

(a) the Lie derivative is independent of the metric,

(b) the Lie derivative, with respect to a vector field $u^{\mathbf{a}}$, acting on a covector field $\omega_{\mathbf{a}}$ is

$$\mathcal{L}_u \omega_{\mathbf{a}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \left(\nabla_{\mathbf{a}} u^{\mathbf{b}}\right) \omega_{\mathbf{b}} \tag{Q4.1.1}$$

(c) Derive Eq. (1.2.9).

(d) Show that a coordinate basis vector $e_\alpha^{\mathbf{a}}$ is a Killing vector if and only if

$$\nabla_\alpha g_{\beta\gamma} = 0 \tag{Q4.1.2}$$

for all β, γ, and explain the difference between $\nabla_\alpha g_{\beta\gamma}$ and $\nabla_{\mathbf{a}} g_{\mathbf{bc}}$.

(e) Show that a particle with momentum

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \tag{Q4.1.3}$$

and moving freely in a space with Killing vector field $\xi^{\mathbf{a}}$ has conserved quantity $\xi^{\mathbf{a}} p_{\mathbf{a}}$.

(f) By considering Cartesian and polar coordinates, determine the conserved quantities of a particle moving freely in two dimensional Euclidean space.

A4.1. (a) Using Eqs. (1.2.8), (1.2.27) and (Q3.1.2), in a coordinate basis,

$$\begin{aligned}
\mathcal{L}_u v^{\mathbf{a}} &= u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} && \text{(A4.1.1)} \\
&= u^\beta \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma \right) e_\alpha^{\mathbf{a}} - v^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma \right) e_\alpha^{\mathbf{a}} && \text{(A4.1.2)} \\
&= \left(u^\beta \frac{\partial v^\alpha}{\partial x^\beta} - v^\beta \frac{\partial u^\alpha}{\partial x^\beta} \right) e_\alpha^{\mathbf{a}} && \text{(A4.1.3)}
\end{aligned}$$

which is independent of the metric.

(b) Using the Leibniz rule and Eq. (1.2.8),

$$\begin{aligned}
v^{\mathbf{a}} \mathcal{L}_u \omega_{\mathbf{a}} &= \mathcal{L}_u \left(\omega_{\mathbf{a}} v^{\mathbf{a}} \right) - \omega_{\mathbf{a}} \mathcal{L}_u v^{\mathbf{a}} && \text{(A4.1.4)} \\
&= u^{\mathbf{b}} \nabla_{\mathbf{b}} \left(\omega_{\mathbf{a}} v^{\mathbf{a}} \right) - \omega_{\mathbf{a}} \left(u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} \right) && \text{(A4.1.5)} \\
&= v^{\mathbf{a}} \left(u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \right) && \text{(A4.1.6)}
\end{aligned}$$

hence Eq. (Q4.1.1).

(c) Using the Leibniz rule and Eq. (1.2.8),

$$\begin{aligned}
u^{\mathbf{a}} v^{\mathbf{b}} \mathcal{L}_\xi g_{\mathbf{ab}} &= \mathcal{L}_\xi \left(u^{\mathbf{a}} v^{\mathbf{b}} g_{\mathbf{ab}} \right) - g_{\mathbf{ab}} v^{\mathbf{b}} \mathcal{L}_\xi u^{\mathbf{a}} - g_{\mathbf{ab}} u^{\mathbf{a}} \mathcal{L}_\xi v^{\mathbf{b}} && \text{(A4.1.7)} \\
&= \xi^{\mathbf{c}} \nabla_{\mathbf{c}} \left(u^{\mathbf{a}} v^{\mathbf{b}} g_{\mathbf{ab}} \right) - g_{\mathbf{ab}} v^{\mathbf{b}} \left(\xi^{\mathbf{c}} \nabla_{\mathbf{c}} u^{\mathbf{a}} - u^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{a}} \right) \\
&\quad - g_{\mathbf{ab}} u^{\mathbf{a}} \left(\xi^{\mathbf{c}} \nabla_{\mathbf{c}} v^{\mathbf{b}} - v^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{b}} \right) && \text{(A4.1.8)} \\
&= g_{\mathbf{ab}} v^{\mathbf{b}} u^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{a}} + g_{\mathbf{ab}} u^{\mathbf{a}} v^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{b}} && \text{(A4.1.9)} \\
&= u^{\mathbf{a}} v^{\mathbf{b}} \left(\nabla_{\mathbf{a}} \xi_{\mathbf{b}} + \nabla_{\mathbf{b}} \xi_{\mathbf{a}} \right) && \text{(A4.1.10)}
\end{aligned}$$

(d) Using Eqs. (1.2.9), (1.2.26) and (1.2.29),

$$\begin{aligned}
\mathcal{L}_{e_\alpha} g_{\mathbf{ab}} &= \nabla_{\mathbf{a}} \left(g_{\mathbf{bc}} e^{\mathbf{c}}_{\alpha} \right) + \nabla_{\mathbf{b}} \left(g_{\mathbf{ac}} e^{\mathbf{c}}_{\alpha} \right) && \text{(A4.1.11)} \\
&= g_{\mathbf{bc}} \Gamma^{\mathbf{c}}_{\mathbf{a}\alpha} + g_{\mathbf{ac}} \Gamma^{\mathbf{c}}_{\mathbf{b}\alpha} && \text{(A4.1.12)} \\
&= \frac{1}{2} e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{a}} \left(g_{\beta\gamma,\alpha} + g_{\beta\alpha,\gamma} - g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} - g_{\beta\alpha,\gamma} \right) && \text{(A4.1.13)} \\
&= e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{a}} \nabla_{\alpha} g_{\beta\gamma} && \text{(A4.1.14)}
\end{aligned}$$

$\nabla_\alpha g_{\beta\gamma}$ is the partial derivative of the metric components with respect to the coordinate x^α and is coordinate dependent, while $\nabla_{\mathbf{a}} g_{\mathbf{bc}}$ is the covariant derivative of the metric tensor and

$$\nabla_{\mathbf{a}} g_{\mathbf{bc}} \equiv 0 \qquad \text{(A4.1.15)}$$

(e) Using Eqs. (Q4.1.3) and (1.2.9),

$$\begin{aligned}
\frac{d}{dt} \left(\xi^{\mathbf{a}} p_{\mathbf{a}} \right) &= m \frac{d}{dt} \left(\xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \right) && \text{(A4.1.16)} \\
&= m \frac{dx^{\mathbf{b}}}{dt} \frac{dx^{\mathbf{c}}}{dt} \nabla_{\mathbf{c}} \left(\xi^{\mathbf{a}} g_{\mathbf{ab}} \right) + m \xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{d^2 x^{\mathbf{b}}}{dt^2} && \text{(A4.1.17)} \\
&= \frac{1}{2} m \frac{dx^{\mathbf{a}}}{dt} \frac{dx^{\mathbf{b}}}{dt} \mathcal{L}_{\xi} g_{\mathbf{ab}} + m \xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{d^2 x^{\mathbf{b}}}{dt^2} && \text{(A4.1.18)}
\end{aligned}$$

(f) From Eq. (1.2.22), two dimensional Euclidean space has translational symmetries generated by $e^{\mathbf{a}}_x$ and $e^{\mathbf{a}}_y$, which give rise to the conserved quantities

$$\begin{aligned}
e^{\mathbf{a}}_x p_{\mathbf{a}} &= p_x = m g_{xx} \dot{x} = m \dot{x} && \text{(A4.1.19)} \\
e^{\mathbf{a}}_y p_{\mathbf{a}} &= p_y = m g_{yy} \dot{y} = m \dot{y} && \text{(A4.1.20)}
\end{aligned}$$

and, from Eq. (1.2.23), has rotational symmetry generated by $e^{\mathbf{a}}_\theta$, which gives rise to the conserved quantity

$$e^{\mathbf{a}}_\theta p_{\mathbf{a}} = p_\theta = m g_{\theta\theta} \dot{\theta} = m r^2 \dot{\theta} \qquad \text{(A4.1.21)}$$