Homework 5 - Torsion

Q5.1. If the torsion is non-zero, Eqs. (1.2.6), (1.2.7) and (1.2.8) become

$$(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}})\,\phi = -T^{\mathbf{c}}{}_{\mathbf{ab}}\nabla_{\mathbf{c}}\phi \tag{Q5.1.1}$$

$$(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}})\,\omega_{\mathbf{c}} = R_{\mathbf{abc}}{}^{\mathbf{d}}\omega_{\mathbf{d}} - T^{\mathbf{d}}{}_{\mathbf{ab}}\nabla_{\mathbf{d}}\omega_{\mathbf{c}} \tag{Q5.1.2}$$

and

$$\mathcal{L}_u v^{\mathbf{a}} = u^{\mathbf{b}}\nabla_{\mathbf{b}}v^{\mathbf{a}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}} - T^{\mathbf{a}}{}_{\mathbf{bc}}u^{\mathbf{b}}v^{\mathbf{c}} \tag{Q5.1.3}$$

see Figure Q5.1.1.

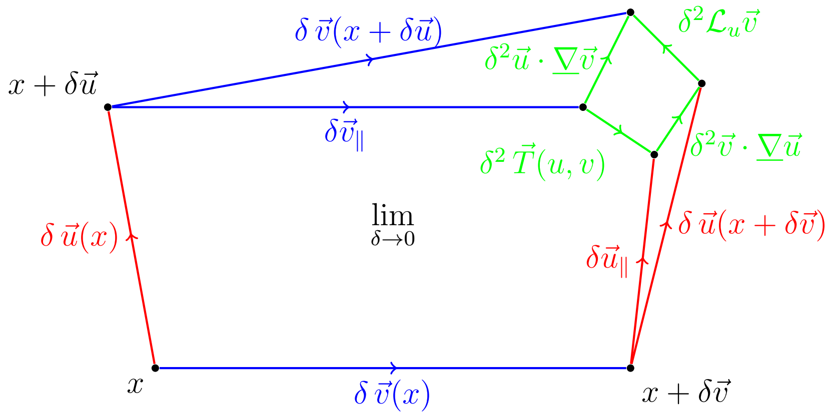

Figure Q5.1.1: The relationship between the Lie derivative, covariant derivative and torsion. Here, $\vec{v}_{\parallel}$ is $\vec{v}(x)$ parallel transported along $\vec{u}$ to $x+\delta\vec{u}$, i.e. transported such that $\vec{u}\cdot\underline{\nabla}\vec{v}=0$, $\vec{u}_{\parallel}$ is $\vec{u}(x)$ parallel transported along $\vec{v}$ to $x+\delta\vec{v}$ and $\vec{T}(u,v)\leftrightarrow T^{\mathbf{a}}{}_{\mathbf{bc}}u^{\mathbf{b}}v^{\mathbf{c}}$.

Show that

(a) if $u^{\mathbf{a}}$ and $v^{\mathbf{a}}$ commute then

$$\left(u^{\mathbf{a}}\nabla_{\mathbf{a}}v^{\mathbf{b}}\nabla_{\mathbf{b}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}}\nabla_{\mathbf{a}}\right)w^{\mathbf{c}} = u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{ab}}{}^{\mathbf{c}}{}_{\mathbf{d}}w^{\mathbf{d}} \tag{Q5.1.4}$$

(b) the Lie derivative is independent of the metric.