Homework 5 - Torsion

Q5.1. If the torsion is non-zero, Eqs. (1.2.6), (1.2.7) and (1.2.8) become

$$(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}})\,\phi = -T^{\mathbf{c}}{}_{\mathbf{ab}}\nabla_{\mathbf{c}}\phi \tag{Q5.1.1}$$

$$(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}})\,\omega_{\mathbf{c}} = R_{\mathbf{abc}}{}^{\mathbf{d}}\omega_{\mathbf{d}} - T^{\mathbf{d}}{}_{\mathbf{ab}}\nabla_{\mathbf{d}}\omega_{\mathbf{c}} \tag{Q5.1.2}$$

and

$$\mathcal{L}_u v^{\mathbf{a}} = u^{\mathbf{b}}\nabla_{\mathbf{b}}v^{\mathbf{a}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}} - T^{\mathbf{a}}{}_{\mathbf{bc}}u^{\mathbf{b}}v^{\mathbf{c}} \tag{Q5.1.3}$$

see Figure Q5.1.1.

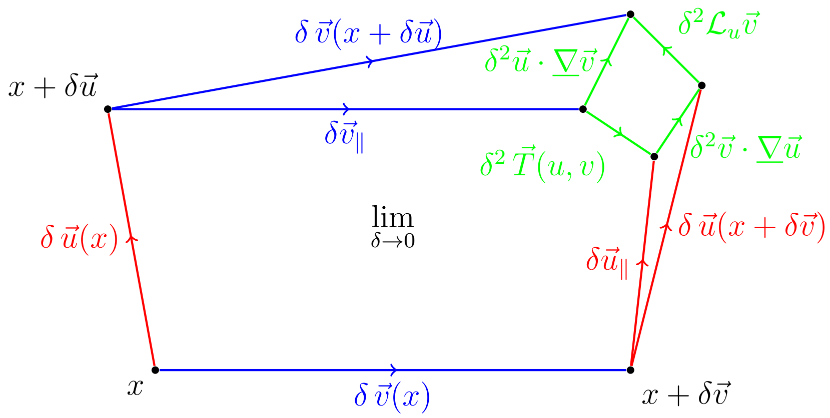

Figure Q5.1.1: The relationship between the Lie derivative, covariant derivative and torsion. Here, $\vec{v}_\parallel$ is $\vec{v}(x)$ parallel transported along $\vec{u}$ to $x+\delta\vec{u}$, i.e. transported such that $\vec{u}\cdot\underline{\nabla}\vec{v}=0$, $\vec{u}_\parallel$ is $\vec{u}(x)$ parallel transported along $\vec{v}$ to $x+\delta\vec{v}$ and $\vec{T}(u,v)\leftrightarrow T^{\mathbf{a}}{}_{\mathbf{bc}}u^{\mathbf{b}}v^{\mathbf{c}}$.

Show that

(a) if $u^{\mathbf{a}}$ and $v^{\mathbf{a}}$ commute then

$$\left(u^{\mathbf{a}}\nabla_{\mathbf{a}}v^{\mathbf{b}}\nabla_{\mathbf{b}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}}\nabla_{\mathbf{a}}\right)w^{\mathbf{c}} = u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{ab}}{}^{\mathbf{c}}{}_{\mathbf{d}}w^{\mathbf{d}} \tag{Q5.1.4}$$

(b) the Lie derivative is independent of the metric.

A5.1. (a) Using Eqs. (Q5.1.2) and (Q5.1.3),

$$\left(u^{\mathbf{a}}\nabla_{\mathbf{a}}v^{\mathbf{b}}\nabla_{\mathbf{b}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}}\nabla_{\mathbf{a}}\right)w^{\mathbf{c}}$$

$$= u^{\mathbf{a}}v^{\mathbf{b}}\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right)w^{\mathbf{c}} + u^{\mathbf{a}}\left(\nabla_{\mathbf{a}}v^{\mathbf{b}}\right)\nabla_{\mathbf{b}}w^{\mathbf{c}} - v^{\mathbf{b}}\left(\nabla_{\mathbf{b}}u^{\mathbf{a}}\right)\nabla_{\mathbf{a}}w^{\mathbf{c}} \tag{A5.1.1}$$

$$= u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{ab}}{}^{\mathbf{c}}{}_{\mathbf{d}}w^{\mathbf{d}} - u^{\mathbf{a}}v^{\mathbf{b}}T^{\mathbf{d}}{}_{\mathbf{ab}}\nabla_{\mathbf{d}}w^{\mathbf{c}} + \left(u^{\mathbf{a}}\nabla_{\mathbf{a}}v^{\mathbf{b}} - v^{\mathbf{a}}\nabla_{\mathbf{a}}u^{\mathbf{b}}\right)\nabla_{\mathbf{b}}w^{\mathbf{c}} \tag{A5.1.2}$$

$$= u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{ab}}{}^{\mathbf{c}}{}_{\mathbf{d}}w^{\mathbf{d}} + \left(\mathcal{L}_u v^{\mathbf{b}}\right)\nabla_{\mathbf{b}}w^{\mathbf{c}} \tag{A5.1.3}$$

$$= u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{ab}}{}^{\mathbf{c}}{}_{\mathbf{d}}w^{\mathbf{d}} \tag{A5.1.4}$$

(b) Using Eqs. (Q3.1.1) and (Q5.1.1), in a coordinate basis,

$$\begin{aligned}
\Gamma^\alpha_{\beta\gamma} e^\beta_{\mathbf{b}} e^\gamma_{\mathbf{c}} &= -\nabla_{\mathbf{b}} e^\alpha_{\mathbf{c}} = -\nabla_{\mathbf{b}} \nabla_{\mathbf{c}} x^\alpha && \text{(A5.1.5)} \\
&= -\nabla_{\mathbf{c}} \nabla_{\mathbf{b}} x^\alpha + T^{\mathbf{a}}{}_{\mathbf{bc}} \nabla_{\mathbf{a}} x^\alpha && \text{(A5.1.6)} \\
&= -\nabla_{\mathbf{c}} e^\alpha_{\mathbf{b}} + T^{\mathbf{a}}{}_{\mathbf{bc}} e^\alpha_{\mathbf{a}} && \text{(A5.1.7)} \\
&= \Gamma^\alpha_{\gamma\beta} e^\gamma_{\mathbf{c}} e^\beta_{\mathbf{b}} + T^\alpha{}_{\beta\gamma} e^\beta_{\mathbf{b}} e^\gamma_{\mathbf{c}} && \text{(A5.1.8)}
\end{aligned}$$

therefore

$$\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} = T^\alpha{}_{\beta\gamma} \qquad \text{(A5.1.9)}$$

Using Eqs. (Q5.1.3), (1.2.27) and (A5.1.9), in a coordinate basis,

$$\begin{aligned}
\mathcal{L}_u v^{\mathbf{a}} &= u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} - T^{\mathbf{a}}{}_{\mathbf{bc}} u^{\mathbf{b}} v^{\mathbf{c}} && \text{(A5.1.10)} \\
&= u^\beta \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma \right) e^{\mathbf{a}}_\alpha - v^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma \right) e^{\mathbf{a}}_\alpha - T^\alpha{}_{\beta\gamma} u^\beta v^\gamma e^{\mathbf{a}}_\alpha && \text{(A5.1.11)} \\
&= \left(u^\beta \frac{\partial v^\alpha}{\partial x^\beta} - v^\beta \frac{\partial u^\alpha}{\partial x^\beta} \right) e^{\mathbf{a}}_\alpha + u^\beta v^\gamma \left(\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} - T^\alpha{}_{\beta\gamma} \right) e^{\mathbf{a}}_\alpha && \text{(A5.1.12)} \\
&= \left(u^\beta \frac{\partial v^\alpha}{\partial x^\beta} - v^\beta \frac{\partial u^\alpha}{\partial x^\beta} \right) e^{\mathbf{a}}_\alpha && \text{(A5.1.13)}
\end{aligned}$$

which is independent of the metric.