Homework 6 - Newtonian perspective

Q6.1. For a particle with spacetime velocity $u^{\mathbf{a}}$ experiencing a spacetime force $f_{\mathbf{a}}$, show that

$$f_{\mathbf{a}} u^{\mathbf{a}} = 0 \tag{Q6.1.1}$$

and interpret this equation from a Newtonian perspective.

A6.1. Using Eqs. (2.3.13) and (2.3.3),

$$f_{\mathbf{a}} u^{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{d\tau} u^{\mathbf{a}} = m g_{\mathbf{ab}} \frac{du^{\mathbf{b}}}{d\tau} u^{\mathbf{a}} = \frac{1}{2} m \frac{d}{d\tau} \left(g_{\mathbf{ab}} u^{\mathbf{a}} u^{\mathbf{b}} \right) = 0 \tag{A6.1.1}$$

Decomposing as in Eqs. (2.3.16) and (2.3.18),

$$\begin{aligned} 0 &= f_{\mathbf{a}} u^{\mathbf{a}} && \text{(A6.1.2)} \\ &= \frac{dt}{d\tau} \left(P e^t_{\mathbf{a}} - F_{\mathbf{a}} \right) \frac{dt}{d\tau} \left(e^{\mathbf{a}}_t + v^{\mathbf{a}} \right) && \text{(A6.1.3)} \\ &= \left(\frac{dt}{d\tau} \right)^2 \left(P - F_{\mathbf{a}} v^{\mathbf{a}} \right) && \text{(A6.1.4)} \end{aligned}$$

Thus Eq. (Q6.1.1) corresponds to the Newtonian work-energy theorem.

Q6.2. In the space-time decomposition of the metric, Eq. (2.2.9), set

$$A = 1 + 2\phi \tag{Q6.2.1}$$

and use the Newtonian approximation

$$\phi \,,\, B_{\mathbf{a}} v^{\mathbf{a}} \,,\, h_{\mathbf{ab}} v^{\mathbf{a}} v^{\mathbf{b}} \ll 1 \tag{Q6.2.2}$$

(a) Reexpress the relativistic uncharged particle action in space-time decomposed form and interpret the leading terms in your action.

(b) Calculate the Euler-Lagrange equation for the leading terms in your action and interpret the resulting equation.

(c) Reexpress the spacetime momentum in space-time decomposed form and interpret the leading terms.

A6.2. (a) Using Eqs. (2.3.10), (2.2.9), (2.3.16) and (Q6.2.1), in the Newtonian limit, the relativistic particle action is

$$\begin{aligned} -S &= m \int \sqrt{g_{\mathbf{ab}} \dot{x}^{\mathbf{a}} \dot{x}^{\mathbf{b}}} \, dt && \text{(A6.2.1)} \\ &= m \int \sqrt{\left(A e^t_{\mathbf{a}} e^t_{\mathbf{b}} - B_{\mathbf{a}} e^t_{\mathbf{b}} - e^t_{\mathbf{a}} B_{\mathbf{b}} - h_{\mathbf{ab}} \right) \left(e^{\mathbf{a}}_t + v^{\mathbf{a}} \right) \left(e^{\mathbf{b}}_t + v^{\mathbf{b}} \right)} \, dt && \text{(A6.2.2)} \\ &= m \int \sqrt{1 + 2\phi - 2 B_{\mathbf{a}} v^{\mathbf{a}} - h_{\mathbf{ab}} v^{\mathbf{a}} v^{\mathbf{b}}} \, dt && \text{(A6.2.3)} \\ &\simeq m \int \left(1 + \phi - B_{\mathbf{a}} v^{\mathbf{a}} - \frac{1}{2} h_{\mathbf{ab}} v^{\mathbf{a}} v^{\mathbf{b}} \right) dt && \text{(A6.2.4)} \end{aligned}$$

We can interpret ϕ as the Newtonian gravitational potential, $B_{\mathbf{a}}$ as the gravitational analog of the magnetic covector potential $A^{(3)}_{\mathbf{a}}$ and $\frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}$ as the Newtonian kinetic energy.

(b) For the action of Eq. (A6.2.4), the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}_{3}}\right) = \frac{\partial L}{\partial x^{\mathbf{a}}_{3}} \tag{A6.2.5}$$

gives

$$m\frac{d}{dt}\left(h_{\mathbf{ab}}v^{\mathbf{b}} + B_{\mathbf{a}}\right) = -m\nabla^{(3)}_{\mathbf{a}}\phi + mv^{\mathbf{b}}\nabla^{(3)}_{\mathbf{a}}B_{\mathbf{b}} \tag{A6.2.6}$$

therefore

$$m\frac{dv_{\mathbf{a}}}{dt} = m\left[\left(-\nabla^{(3)}_{\mathbf{a}}\phi - \dot{B}_{\mathbf{a}}\right) + \left(\nabla^{(3)}_{\mathbf{a}}B_{\mathbf{b}} - \nabla^{(3)}_{\mathbf{b}}B_{\mathbf{a}}\right)v^{\mathbf{b}}\right] \tag{A6.2.7}$$

We can interpret $-\nabla^{(3)}_{\mathbf{a}}\phi$ as the Newtonian gravitational field analogous to the electrostatic field, $-\nabla^{(3)}_{\mathbf{a}}\phi - \dot{B}_{\mathbf{a}}$ as the gravitational analog of the electric field $E_{\mathbf{a}} = -\nabla^{(3)}_{\mathbf{a}}\phi - \dot{A}^{(3)}_{\mathbf{a}}$, and $\nabla^{(3)}_{\mathbf{a}}B_{\mathbf{b}} - \nabla^{(3)}_{\mathbf{b}}B_{\mathbf{a}}$ as the gravitational analog of the magnetic flux density $B_{\mathbf{ab}} = \nabla^{(3)}_{\mathbf{a}}A^{(3)}_{\mathbf{b}} - \nabla^{(3)}_{\mathbf{b}}A^{(3)}_{\mathbf{a}}$. The whole equation is thus the gravitational analog of the Lorentz force law Eq. (2.3.26).

(c) The spacetime momentum is decomposed in terms of the energy and spatial momentum as

$$p_{\mathbf{a}} = Ee^{t}_{\mathbf{a}} - p^{(3)}_{\mathbf{a}} \tag{A6.2.8}$$

Substituting Eqs. (2.2.9), (2.3.16) and (Q6.2.1) into Eq. (2.3.13) gives

$$\begin{aligned} p_{\mathbf{a}} &= mg_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} && \text{(A6.2.9)}\\ &= m\frac{dt}{d\tau}\left(Ae^{t}_{\mathbf{a}}e^{t}_{\mathbf{b}} - B_{\mathbf{a}}e^{t}_{\mathbf{b}} - e^{t}_{\mathbf{a}}B_{\mathbf{b}} - h_{\mathbf{ab}}\right)\left(e^{\mathbf{b}}_{t} + v^{\mathbf{b}}\right) && \text{(A6.2.10)}\\ &= m\frac{dt}{d\tau}\left[\left(1 + 2\phi - B_{\mathbf{b}}v^{\mathbf{b}}\right)e^{t}_{\mathbf{a}} - \left(B_{\mathbf{a}} + h_{\mathbf{ab}}v^{\mathbf{b}}\right)\right] && \text{(A6.2.11)} \end{aligned}$$

and, from Eq. (A6.2.3),

$$\frac{d\tau}{dt} = \sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}} \tag{A6.2.12}$$

Therefore, comparing Eqs. (A6.2.8) and (A6.2.11), in the Newtonian limit,

$$\begin{aligned} E &= \frac{m\left(1 + 2\phi - B_{\mathbf{a}}v^{\mathbf{a}}\right)}{\sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}}} && \text{(A6.2.13)}\\ &\simeq m + m\phi + \frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}} && \text{(A6.2.14)} \end{aligned}$$

We can interpret m as the mass energy, $m\phi$ as the Newtonian gravitational potential energy and $\frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}$ as the Newtonian kinetic energy. Also

$$p^{(3)}_{\mathbf{a}} = \frac{m\left(B_{\mathbf{a}} + h_{\mathbf{ab}}v^{\mathbf{b}}\right)}{\sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}}} \tag{A6.2.15}$$

$$\simeq mh_{\mathbf{ab}}v^{\mathbf{b}} + mB_{\mathbf{a}} \tag{A6.2.16}$$

We can interpret $mh_{\mathbf{ab}}v^{\mathbf{b}}$ as the Newtonian momentum and $mB_{\mathbf{a}}$ as the gravitational momentum analogous to the electromagnetic momentum $qA^{(3)}_{\mathbf{a}}$.