Homework 7 - Einstein equation

Q7.1. Use the Bianchi identity

$$\nabla_{\mathbf{a}} R_{\mathbf{bcd}}{}^{\mathbf{e}} + \nabla_{\mathbf{b}} R_{\mathbf{cad}}{}^{\mathbf{e}} + \nabla_{\mathbf{c}} R_{\mathbf{abd}}{}^{\mathbf{e}} = 0 \tag{Q7.1.1}$$

to show that

$$\nabla_{\mathbf{b}} G_{\mathbf{a}}{}^{\mathbf{b}} = 0 \tag{Q7.1.2}$$

A7.1. Contracting the **c** and **e** indices, contracting the **a** and **d** indices using the metric, using the symmetries Eqs. (A1.1.6) and (Q1.1.2) and the definitions Eqs (A1.1.8), (A1.1.9) and (2.4.7) gives Eq. (Q7.1.2)

$$\begin{aligned} 0 &= g^{\mathbf{ad}} \nabla_{\mathbf{a}} R_{\mathbf{bcd}}{}^{\mathbf{c}} + g^{\mathbf{ad}} \nabla_{\mathbf{b}} R_{\mathbf{cad}}{}^{\mathbf{c}} + g^{\mathbf{ad}} \nabla_{\mathbf{c}} R_{\mathbf{abd}}{}^{\mathbf{c}} && \text{(A7.1.1)} \\ &= \nabla_{\mathbf{a}} R_{\mathbf{bc}}{}^{\mathbf{ac}} - \nabla_{\mathbf{b}} R_{\mathbf{ac}}{}^{\mathbf{ac}} + \nabla_{\mathbf{c}} R_{\mathbf{ba}}{}^{\mathbf{ca}} && \text{(A7.1.2)} \\ &= 2 \nabla_{\mathbf{a}} R_{\mathbf{b}}{}^{\mathbf{a}} - \nabla_{\mathbf{b}} R && \text{(A7.1.3)} \\ &= 2 \nabla_{\mathbf{a}} G_{\mathbf{b}}{}^{\mathbf{a}} && \text{(A7.1.4)} \end{aligned}$$

Q7.2. Calculate

$$\nabla_{\mathbf{b}} T_{\mathbf{a}}{}^{\mathbf{b}} = 0 \tag{Q7.2.1}$$

for a perfect fluid and interpret your answer.

A7.2. Eq. (2.4.13) gives

$$\begin{aligned} 0 = \nabla_{\mathbf{b}} T_{\mathbf{a}}{}^{\mathbf{b}} &= \nabla_{\mathbf{b}} \left[\rho u_{\mathbf{a}} u^{\mathbf{b}} + p \left(u_{\mathbf{a}} u^{\mathbf{b}} - \delta_{\mathbf{a}}^{\mathbf{b}} \right) \right] && \text{(A7.2.1)} \\ &= u_{\mathbf{a}} u^{\mathbf{b}} \nabla_{\mathbf{b}} \rho + (\rho + p)\, u^{\mathbf{b}} \nabla_{\mathbf{b}} u_{\mathbf{a}} + (\rho + p)\, u_{\mathbf{a}} \nabla_{\mathbf{b}} u^{\mathbf{b}} \\ &\quad + \left(u_{\mathbf{a}} u^{\mathbf{b}} - \delta_{\mathbf{a}}^{\mathbf{b}} \right) \nabla_{\mathbf{b}} p && \text{(A7.2.2)} \end{aligned}$$

therefore, using Eqs. (2.3.3) and (2.3.4),

$$u^{\mathbf{a}} \nabla_{\mathbf{b}} T_{\mathbf{a}}{}^{\mathbf{b}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} \rho + (\rho + p)\, \nabla_{\mathbf{b}} u^{\mathbf{b}} = 0 \tag{A7.2.3}$$

and

$$\left(g^{\mathbf{ac}} - u^{\mathbf{a}} u^{\mathbf{c}} \right) \nabla_{\mathbf{b}} T_{\mathbf{c}}{}^{\mathbf{b}} = (\rho + p)\, u^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} + \left(u^{\mathbf{a}} u^{\mathbf{b}} - g^{\mathbf{ab}} \right) \nabla_{\mathbf{b}} p = 0 \tag{A7.2.4}$$

which are the relativistic Euler equations, corresponding to energy and momentum conservation in the rest frame of the fluid. Writing the energy density ρ in terms of the mass density ρ_m and non-mass energy per unit mass e

$$\rho = \rho_m (1 + e) \tag{A7.2.5}$$

Eq. (A7.2.3) gives

$$(1 + e) \left(u^{\mathbf{b}} \nabla_{\mathbf{b}} \rho_m + \rho_m \nabla_{\mathbf{b}} u^{\mathbf{b}} \right) + \left(\rho_m u^{\mathbf{b}} \nabla_{\mathbf{b}} e + p \nabla_{\mathbf{b}} u^{\mathbf{b}} \right) = 0 \tag{A7.2.6}$$

corresponding to conservation of mass and Newtonian energy.