Homework 9 - Schwarzschild metric

Q9.1. You synchronize two clocks and then throw one up into the air and catch it. Which clock will show the earlier time or will they show the same time?

A9.1. Extremizing the action for a free particle

$$S = -m \int d\tau \tag{A9.1.1}$$

see Eq. (2.3.5), maximizes the length (proper time interval) of the worldline, since null deformations reduce the length of a timelike curve. Thus the freely moving clock, i.e. the thrown clock, will show the later time and the held clock will show the earlier time.

We can see how this works in Schwarzschild coordinates. For radial motion, Eq (2.6.2) gives

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{r} - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2} \tag{A9.1.2}$$

Taking $r \simeq r_0 \gg GM$ and using the Newtonian solution of Eqs. (2.6.9) and (2.6.13)

$$r \simeq r_0 + v_0 t - \frac{1}{2} g_0 t^2 \tag{A9.1.3}$$

with

$$g_0 = \frac{GM}{r_0^2} \tag{A9.1.4}$$

gives

$$\frac{d\tau}{dt} \simeq 1 - r_0 g_0 + g_0 \left(v_0 t - \frac{1}{2} g_0 t^2\right) - \frac{1}{2} (v_0 - g_0 t)^2 \tag{A9.1.5}$$

Integrating and evaluating when the clock returns at $t \simeq 2v_0/g_0$ gives

$$\tau_{\text{thrown}} \simeq \frac{2v_0}{g_0} - 2r_0 v_0 + \frac{2}{3}\frac{v_0^3}{g_0} - \frac{1}{6}\frac{v_0^3}{g_0} \tag{A9.1.6}$$

for the thrown clock compared with

$$\tau_{\text{held}} \simeq \frac{2v_0}{g_0} - 2r_0 v_0 \tag{A9.1.7}$$

for the held clock. Thus the reduction in the gravitional time dilation for the thrown clock is four times its velocity time dilation and so the held clock shows a time $v_0^3/2g_0$ earlier than the thrown clock.

Q9.2. A particle of mass m falls towards a Schwarzschild black hole of mass M. The particle is initially a great distance from the black hole and has energy $E = m$ and angular momentum L slightly greater than $4GMm$. Describe qualitatively its trajectory.

Another particle falls under the same conditions except that its angular momentum is slightly smaller than $4GMm$. Describe qualitatively its trajectory.

A9.2. For $E = m$ and $L = 4GMm$, Eqs. (2.6.13) and (2.6.14) become

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V(r) = 0 \tag{A9.2.1}$$

with

$$V(r) = -\frac{GMm}{r}\left(1 - \frac{4GM}{r}\right)^2 \tag{A9.2.2}$$

$r = 4GM$ corresponds to an unstable circular orbit and, since $V(4GM) = V(\infty)$, a particle falling in from infinity will asymptotically approach this orbit.

In the case of L slightly greater than $4GMm$, the angular momentum barrier will be slightly higher and the particle will approach $r = 4GM$, orbiting there for some period of time, before eventually escaping back to infinity. In the case of L slightly less than $4GMm$, the angular momentum barrier will be slightly lower and the particle will approach $r = 4GM$, orbiting there for some period of time, before eventually falling into the black hole.