Homework 10 - Weyl tensor

Q10.1. Show that

$$C_{\mathbf{abc}}{}^{\mathbf{b}} = 0 \tag{Q10.1.1}$$

A10.1. Contracting the **b** and **d** indices in Eq. (2.7.1) with the metric gives

$$\begin{aligned} R_{\mathbf{ac}} &= C_{\mathbf{abc}}{}^{\mathbf{b}} + \frac{1}{N-2}\left(NR_{\mathbf{ac}} - R_{\mathbf{ac}} - R_{\mathbf{ac}} + Rg_{\mathbf{ac}}\right) \\ &\quad - \frac{1}{(N-1)(N-2)}R\left(Ng_{\mathbf{ac}} - g_{\mathbf{ac}}\right) \end{aligned} \tag{A10.1.1}$$

and hence Eq. (Q10.1.1).

Q10.2. Show that in four dimensional spacetime

$$\nabla_{\mathbf{d}} C_{\mathbf{abc}}{}^{\mathbf{d}} = -\nabla_{[\mathbf{a}} D_{\mathbf{b}]\mathbf{c}} \tag{Q10.2.1}$$

where

$$D_{\mathbf{bc}} = R_{\mathbf{bc}} - \frac{1}{6}Rg_{\mathbf{bc}} \tag{Q10.2.2}$$

A10.2. Contracting the **a** and **e** indices in Eq. (Q7.1.1) gives

$$0 = \nabla_{\mathbf{d}} R_{\mathbf{abc}}{}^{\mathbf{d}} + \nabla_{\mathbf{a}} R_{\mathbf{bdc}}{}^{\mathbf{d}} + \nabla_{\mathbf{b}} R_{\mathbf{dac}}{}^{\mathbf{d}} \tag{A10.2.1}$$

$$= \nabla_{\mathbf{d}} R_{\mathbf{abc}}{}^{\mathbf{d}} + 2\nabla_{[\mathbf{a}} R_{\mathbf{b}]\mathbf{c}} \tag{A10.2.2}$$

Taking the divergence of Eq. (2.7.1) and using Eq. (A7.1.3) gives

$$\begin{aligned} \nabla_{\mathbf{d}} R_{\mathbf{abc}}{}^{\mathbf{d}} &= \nabla_{\mathbf{d}} C_{\mathbf{abc}}{}^{\mathbf{d}} + \frac{1}{N-2}\left(\nabla_{\mathbf{b}} R_{\mathbf{ac}} - \nabla_{\mathbf{d}} R^{\mathbf{d}}_{\mathbf{a}} g_{\mathbf{bc}} - \nabla_{\mathbf{a}} R_{\mathbf{bc}} + \nabla_{\mathbf{d}} R^{\mathbf{d}}_{\mathbf{b}} g_{\mathbf{ac}}\right) \\ &\quad - \frac{1}{(N-1)(N-2)}\left(\nabla_{\mathbf{b}} R g_{\mathbf{ac}} - \nabla_{\mathbf{a}} R g_{\mathbf{bc}}\right) \end{aligned} \tag{A10.2.3}$$

$$= \nabla_{\mathbf{d}} C_{\mathbf{abc}}{}^{\mathbf{d}} - \frac{2}{N-2}\nabla_{[\mathbf{a}} R_{\mathbf{b}]\mathbf{c}} - \frac{N-3}{(N-1)(N-2)}\nabla_{[\mathbf{a}} R g_{\mathbf{b}]\mathbf{c}} \tag{A10.2.4}$$

Combining Eqs. (A10.2.2) and (A10.2.4) gives

$$\nabla_{\mathbf{d}} C_{\mathbf{abc}}{}^{\mathbf{d}} = -\frac{2(N-3)}{N-2}\nabla_{[\mathbf{a}}\left(R_{\mathbf{b}]\mathbf{c}} - \frac{1}{2(N-1)}Rg_{\mathbf{b}]\mathbf{c}}\right) \tag{A10.2.5}$$

which reduces to Eqs. (Q10.2.1) and (Q10.2.2) in four dimensions. This is the evolution equation for gravitational waves.

Q10.3. Show that

(a) in one dimension

$$R_{\mathbf{abcd}} = 0 \tag{Q10.3.1}$$

(b) in two or less dimensions

$$G_{\mathbf{ab}} = 0 \tag{Q10.3.2}$$

(c) in three or less dimensions

$$C_{\mathbf{abcd}} = 0 \tag{Q10.3.3}$$

and interpret these results physically.

A10.3. (a) The symmetries of the curvature tensor, Eqs. (A1.1.6) or (Q1.1.2), imply

$$R_{0000} = 0 \tag{A10.3.1}$$

and so in one dimension the curvature tensor is identically zero. Therefore the Ricci tensor and scalar are also identically zero. This means that in one dimension there is no curvature.

(b) In two dimensions, the symmetries of the curvature tensor, Eqs. (A1.1.6), (Q1.1.2) and (Q1.2.1), imply it has a single independent component

$$R_{0101} = -R_{0110} = -R_{1001} = R_{1010} \tag{A10.3.2}$$

Therefore, working in an orthogonal basis so that the metric components are diagonal, the Ricci tensor also has one independent component

$$R_{00} = g^{00} R_{0000} + g^{11} R_{0101} = g^{11} R_{0101} \tag{A10.3.3}$$

$$R_{01} = g^{00} R_{0010} + g^{11} R_{0111} = 0 \tag{A10.3.4}$$

$$R_{11} = g^{00} R_{1010} + g^{11} R_{1111} = g^{00} R_{0101} \tag{A10.3.5}$$

and the Ricci scalar

$$R = g^{00} R_{00} + g^{11} R_{11} = 2 g^{00} g^{11} R_{0101} \tag{A10.3.6}$$

therefore

$$R_{00} = \frac{1}{2} R g_{00} \tag{A10.3.7}$$

$$R_{01} = \frac{1}{2} R g_{01} \tag{A10.3.8}$$

$$R_{11} = \frac{1}{2} R g_{11} \tag{A10.3.9}$$

therefore

$$R_{\mathbf{ab}} = \frac{1}{2} R g_{\mathbf{ab}} \tag{A10.3.10}$$

and hence Eq. (Q10.3.2). This means that in two dimensions there is no gravity.

(c) In three dimensions, the symmetries of the curvature tensor, Eqs. (A1.1.6), (Q1.1.2), (Q1.2.1) and (Q1.2.2) imply it has independent components

$$R_{0101}, R_{1212}, R_{2020}, R_{0212}, R_{1020}, R_{2101} \quad \text{(A10.3.11)}$$

but the Weyl tensor also satisfies Eq. (Q10.1.1) and so

$$0 = g^{00}C_{0000} + g^{11}C_{0101} + g^{22}C_{0202} = g^{11}C_{0101} + g^{22}C_{2020} \quad \text{(A10.3.12)}$$

$$0 = g^{00}C_{1010} + g^{11}C_{1111} + g^{22}C_{1212} = g^{00}C_{0101} + g^{22}C_{1212} \quad \text{(A10.3.13)}$$

$$0 = g^{00}C_{2020} + g^{11}C_{2121} + g^{22}C_{2222} = g^{00}C_{2020} + g^{11}C_{1212} \quad \text{(A10.3.14)}$$

$$0 = g^{00}C_{0010} + g^{11}C_{0111} + g^{22}C_{0212} = g^{22}C_{0212} \quad \text{(A10.3.15)}$$

$$0 = g^{00}C_{1020} + g^{11}C_{1121} + g^{22}C_{1222} = g^{00}C_{1020} \quad \text{(A10.3.16)}$$

$$0 = g^{00}C_{2000} + g^{11}C_{2101} + g^{22}C_{2202} = g^{11}C_{2101} \quad \text{(A10.3.17)}$$

therefore all components are zero and hence Eq. (Q10.3.3). This means that in three dimensions there are no gravitational waves, i.e. gravity is not dynamical.