

## Homework 11 - Expanding universe

Q11.1. Calculate the time  $t$  as a function of the scale factor  $a$  for a universe dominated by radiation

$$p_r = \frac{1}{3}\rho_r \quad (\text{Q11.1.1})$$

then matter

$$p_m = 0 \quad (\text{Q11.1.2})$$

then vacuum energy

$$p_v = -\rho_v \quad (\text{Q11.1.3})$$

including the transitions between these eras, and show that your answer reduces to

$$a_r \simeq \left(\frac{4\rho_r a^4}{3}\right)^{\frac{1}{4}} t^{\frac{1}{2}} \quad (\text{Q11.1.4})$$

$$a_m \simeq \left(\frac{3\rho_m a^3}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}} \quad (\text{Q11.1.5})$$

$$a_v \simeq \left(\frac{\rho_m a^3}{4\rho_v}\right)^{\frac{1}{3}} \exp\left(\sqrt{\frac{\rho_v}{3}} t\right) \quad (\text{Q11.1.6})$$

when the respective individual components dominate.

A11.1. Eqs. (3.1.7) and (3.1.10) and Eqs. (Q11.1.1), (Q11.1.2) and (Q11.1.3) give

$$\rho_r \propto a^{-4} \quad (\text{A11.1.1})$$

$$\rho_m \propto a^{-3} \quad (\text{A11.1.2})$$

$$\rho_v \propto a^0 \quad (\text{A11.1.3})$$

and so Eq. (3.1.4) gives

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho = (\rho_r a^4) a^{-4} + (\rho_m a^3) a^{-3} + \rho_v \quad (\text{A11.1.4})$$

$$= Aa^{-4} + Ba^{-3} + C \quad (\text{A11.1.5})$$

Therefore

$$\frac{1}{\sqrt{3}} \int_0^t dt = \int_0^a \frac{da}{a\sqrt{Aa^{-4} + Ba^{-3} + C}} \quad (\text{A11.1.6})$$

$$= \int \frac{a da}{\sqrt{A + Ba + Ca^4}} \quad (\text{A11.1.7})$$

$$\simeq \begin{cases} \int_0^a \frac{a da}{\sqrt{A + Ba}} & \text{for } a \leq a_* \\ \int_0^{a_*} \frac{a da}{\sqrt{A + Ba}} + \int_{a_*}^a \frac{a da}{\sqrt{Ba + Ca^4}} & \text{for } a \geq a_* \end{cases} \quad (\text{A11.1.8})$$

where

$$a_* \gg \frac{A}{B} \Leftrightarrow \rho_m(a_*) \gg \rho_r(a_*) \quad (\text{A11.1.9})$$

$$a_*^3 \ll \frac{B}{C} \Leftrightarrow \rho_m(a_*) \gg \rho_v \quad (\text{A11.1.10})$$

i.e.  $t_*$  is during matter domination.

Therefore for  $a \leq a_*$ , i.e. during radiation and matter domination,

$$\frac{1}{\sqrt{3}}t = \int_0^a \frac{a da}{\sqrt{A + Ba}} \quad (\text{A11.1.11})$$

$$= \frac{1}{B} \int_0^a \frac{(A + Ba - A) da}{\sqrt{A + Ba}} \quad (\text{A11.1.12})$$

$$= \frac{2}{3B^2} \left[ (A + Ba)^{3/2} - 3A(A + Ba)^{1/2} + 2A^{3/2} \right] \quad (\text{A11.1.13})$$

$$= \frac{2}{3B^2} \left( \sqrt{A + Ba} + 2\sqrt{A} \right) \left( \sqrt{A + Ba} - \sqrt{A} \right)^2 \quad (\text{A11.1.14})$$

$$\sim \begin{cases} \frac{a^2}{2A^{1/2}} & \text{for } a \ll \frac{A}{B} \Leftrightarrow \rho_r \gg \rho_m \\ \frac{2a^{3/2}}{3B^{1/2}} & \text{for } a \gg \frac{A}{B} \Leftrightarrow \rho_m \gg \rho_r \end{cases} \quad (\text{A11.1.15})$$

or

$$a \sim \begin{cases} \left( \frac{4A}{3} \right)^{\frac{1}{4}} t^{\frac{1}{2}} & \text{for } a \ll \frac{A}{B} \Leftrightarrow \rho_r \gg \rho_m \\ \left( \frac{3}{4}B \right)^{\frac{1}{3}} t^{\frac{2}{3}} & \text{for } a \gg \frac{A}{B} \Leftrightarrow \rho_m \gg \rho_r \end{cases} \quad (\text{A11.1.16})$$

Also for  $a \geq a_*$ , i.e. during matter and vacuum domination,

$$\frac{1}{\sqrt{3}}(t - t_*) = \int_{a_*}^a \frac{a da}{\sqrt{Ba + Ca^4}} \quad (\text{A11.1.17})$$

$$= \frac{1}{3} \int_{a_*}^a \frac{3a^2 da}{\sqrt{Ba^3 + Ca^6}} \quad (\text{A11.1.18})$$

$$= \frac{1}{3\sqrt{C}} \int_{a_*}^a \frac{3a^2 da}{\sqrt{\left(\frac{B}{2C} + a^3\right)^2 - \left(\frac{B}{2C}\right)^2}} \quad (\text{A11.1.19})$$

$$= \frac{1}{3\sqrt{C}} \left[ \cosh^{-1} \left( 1 + \frac{2C}{B} a^3 \right) - \cosh^{-1} \left( 1 + \frac{2C}{B} a_*^3 \right) \right] \quad (\text{A11.1.20})$$

$$\simeq \frac{1}{3\sqrt{C}} \cosh^{-1} \left( 1 + \frac{2C}{B} a^3 \right) - \frac{2a_*^{3/2}}{3B^{1/2}} \quad (\text{A11.1.21})$$

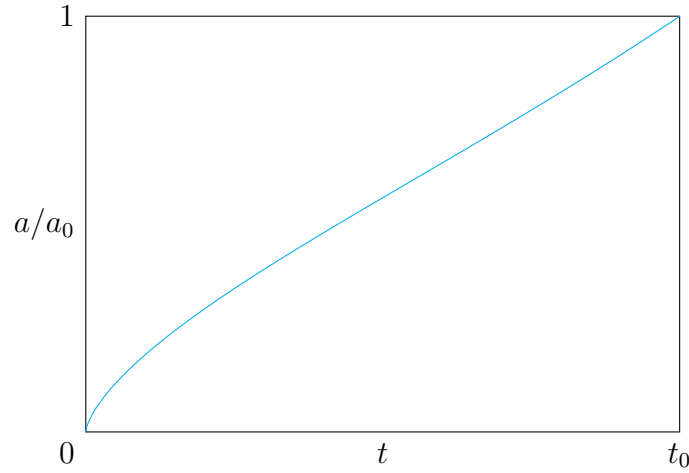


Figure A11.1.1: The expansion of the universe is just beginning to accelerate.

therefore

$$\frac{1}{\sqrt{3}}t = \frac{1}{3\sqrt{C}} \cosh^{-1} \left( 1 + \frac{2C}{B} a^3 \right) \quad (\text{A11.1.22})$$

$$\sim \begin{cases} \frac{2a^{3/2}}{3B^{1/2}} & \text{for } a^3 \ll \frac{B}{C} \Leftrightarrow \rho_m \gg \rho_v \\ \frac{1}{3\sqrt{C}} \ln \left( \frac{4C}{B} a^3 \right) & \text{for } a^3 \gg \frac{B}{C} \Leftrightarrow \rho_v \gg \rho_m \end{cases} \quad (\text{A11.1.23})$$

or

$$a = \left\{ \frac{B}{2C} \left[ \cosh(\sqrt{3C}t) - 1 \right] \right\}^{\frac{1}{3}} \quad (\text{A11.1.24})$$

$$\sim \begin{cases} \left( \frac{3}{4} B t^2 \right)^{\frac{1}{3}} & \text{for } a^3 \ll \frac{B}{C} \Leftrightarrow \rho_m \gg \rho_v \\ \left( \frac{B}{4C} \right)^{\frac{1}{3}} \exp \left( \sqrt{\frac{C}{3}} t \right) & \text{for } a^3 \gg \frac{B}{C} \Leftrightarrow \rho_v \gg \rho_m \end{cases} \quad (\text{A11.1.25})$$

Combining Eqs. (A11.1.14) and (A11.1.22) gives

$$t = \begin{cases} \frac{2}{\sqrt{3}\rho_m^2} (\sqrt{\rho_r + \rho_m} + 2\sqrt{\rho_r}) (\sqrt{\rho_r + \rho_m} - \sqrt{\rho_r})^2 & \text{for } \rho_r + \rho_m \gg \rho_v \\ \frac{1}{\sqrt{3}\rho_v} \cosh^{-1} \left( 1 + \frac{2\rho_v}{\rho_m} \right) & \text{for } \rho_m + \rho_v \gg \rho_r \end{cases} \quad (\text{A11.1.26})$$

during radiation and matter domination and matter and vacuum domination, respectively, and Eqs. (A11.1.16) and (A11.1.25) give Eqs. (Q11.1.4), (Q11.1.5) and (Q11.1.6). See Figure A11.1.1.