Homework 12 - Quantum fields in curved spacetime

Q12.1. Show that if

$$\varphi(\eta,\vec{x}) = \int \frac{d^3\underline{k}}{(2\pi)^{3/2}} \left[a_{\underline{k}}\,\varphi_k(\eta) + a^\dagger_{-\underline{k}}\,\varphi^*_k(\eta)\right] e^{i\underline{k}\cdot\vec{x}} \qquad \text{(Q12.1.1)}$$

with φ_k normalised such that

$$\varphi_k\varphi_k^{*\prime} - \varphi_k'\varphi_k^* = i \qquad \text{(Q12.1.2)}$$

then the commutation relations

$$\begin{aligned} \left[\varphi(\eta,\vec{x})\,,\varphi(\eta,\vec{y})\right] &= 0 & \text{(Q12.1.3)}\\ \left[\varphi(\eta,\vec{x})\,,\varphi'(\eta,\vec{y})\right] &= i\,\delta^3(\vec{x}-\vec{y}) & \text{(Q12.1.4)}\\ \left[\varphi'(\eta,\vec{x})\,,\varphi'(\eta,\vec{y})\right] &= 0 & \text{(Q12.1.5)} \end{aligned}$$

imply

$$\left[a_{\underline{k}}, a_{\underline{l}}\right] = 0 \quad , \quad \left[a_{\underline{k}}, a^\dagger_{\underline{l}}\right] = i\,\delta^3(\vec{x}-\vec{y}) \quad , \quad \left[a^\dagger_{\underline{k}}, a^\dagger_{\underline{l}}\right] = 0 \qquad \text{(Q12.1.6)}$$

A12.1. The Fourier transform of $\varphi(\eta,\vec{x})$ is

$$\begin{aligned} \varphi(\eta,\underline{k}) &\equiv \int \frac{d^3\vec{x}}{(2\pi)^{3/2}}\,\varphi(\eta,\vec{x})\,e^{-i\underline{k}\cdot\vec{x}} & \text{(A12.1.1)}\\ &= a_{\underline{k}}\,\varphi_k(\eta) + a^\dagger_{-\underline{k}}\,\varphi^*_k(\eta) & \text{(A12.1.2)}\\ &= \varphi^\dagger(\eta,-\underline{k}) & \text{(A12.1.3)} \end{aligned}$$

and the Fourier transforms of Eqs. (Q12.1.3), (Q12.1.4) and (Q12.1.5) are

$$\begin{aligned} \left[\varphi(\eta,\underline{k})\,,\varphi^\dagger(\eta,\underline{l})\right] &= 0 & \text{(A12.1.4)}\\ \left[\varphi(\eta,\underline{k})\,,\varphi^{\dagger\prime}(\eta,\underline{l})\right] &= i\,\delta^3(\underline{k}-\underline{l}) & \text{(A12.1.5)}\\ \left[\varphi'(\eta,\underline{k})\,,\varphi^{\dagger\prime}(\eta,\underline{l})\right] &= 0 & \text{(A12.1.6)} \end{aligned}$$

Inverting Eq. (A12.1.2) gives

$$a_{\underline{k}} = -i\left[\varphi_k^{*\prime}(\eta)\,\varphi(\eta,\underline{k}) - \varphi^*_k(\eta)\,\varphi'(\eta,\underline{k})\right] \qquad \text{(A12.1.7)}$$

therefore, using Eqs. (Q12.1.2) and (A12.1.3),

$$\begin{aligned} \left[a_{\underline{k}}, a_{\underline{l}}\right] &= -\left[\varphi_k^{*\prime}(\eta)\,\varphi(\eta,\underline{k}) - \varphi^*_k(\eta)\,\varphi'(\eta,\underline{k})\,,\varphi_l^{*\prime}(\eta)\,\varphi(\eta,\underline{l}) - \varphi^*_l(\eta)\,\varphi'(\eta,\underline{l})\right]\\ &= 0 & \text{(A12.1.8)}\\ \left[a_{\underline{k}}, a^\dagger_{\underline{l}}\right] &= \left[\varphi_k^{*\prime}(\eta)\,\varphi(\eta,\underline{k}) - \varphi^*_k(\eta)\,\varphi'(\eta,\underline{k})\,,\varphi_l'(\eta)\,\varphi^\dagger(\eta,\underline{l}) - \varphi_l(\eta)\,\varphi^{\dagger\prime}(\eta,\underline{l})\right]\\ &= -\,\varphi_k^{*\prime}(\eta)\,\varphi_l(\eta)\left[\varphi(\eta,\underline{k})\,,\varphi^{\dagger\prime}(\eta,\underline{l})\right] - \varphi^*_k(\eta)\,\varphi_l'(\eta)\left[\varphi'(\eta,\underline{k})\,,\varphi^\dagger(\eta,\underline{l})\right]\\ &= \delta^3(\underline{k}-\underline{l}) & \text{(A12.1.9)}\\ \left[a^\dagger_{\underline{k}}, a^\dagger_{\underline{l}}\right] &= -\left[\varphi_k'(\eta)\,\varphi^\dagger(\eta,\underline{k}) - \varphi_k(\eta)\,\varphi^{\dagger\prime}(\eta,\underline{k})\,,\varphi_l'(\eta)\,\varphi^\dagger(\eta,\underline{l}) - \varphi_l(\eta)\,\varphi^{\dagger\prime}(\eta,\underline{l})\right]\\ &= 0 & \text{(A12.1.10)} \end{aligned}$$