

## 2.3 Particles in spacetime

A **particle** is something that exists as a **worldline** in spacetime.

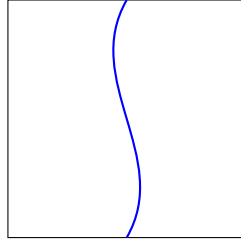


Figure 2.3.1: A **particle** in spacetime.

### 2.3.1 Kinematics

We can parameterize a particle's worldline by its length to define a physical velocity

$$u^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{d\tau} \quad (2.3.1)$$

and acceleration

$$a^{\mathbf{a}} = \frac{d^2x^{\mathbf{a}}}{d\tau^2} \quad (2.3.2)$$

The magnitude of the velocity

$$g_{\mathbf{ab}}u^{\mathbf{a}}u^{\mathbf{b}} = \frac{g_{\mathbf{ab}}dx^{\mathbf{a}}dx^{\mathbf{b}}}{d\tau^2} = 1 \quad (2.3.3)$$

and hence the acceleration is always perpendicular to the velocity

$$g_{\mathbf{ab}}u^{\mathbf{a}}a^{\mathbf{b}} = 0 \quad (2.3.4)$$

### 2.3.2 Action

A worldline  $C$  in a spacetime  $M$  has action

$$-S[C] = \int_C (m\underline{\sigma} + q\underline{A}) \quad (2.3.5)$$

where the worldline volume form  $\underline{\sigma}$  measures the length along the curve

$$d\tau = \underline{\sigma} \cdot \vec{dx} \quad (2.3.6)$$

and  $\underline{A}$  is a covector field in the spacetime. Note that the physics given by  $\delta S = 0$  is invariant under

$$\underline{A} \rightarrow \underline{A} + \underline{\nabla}\lambda \quad (2.3.7)$$

since

$$\int_C \underline{\nabla} \lambda = \int_{\partial C} \lambda \quad (2.3.8)$$

is a boundary term. In Lagrangian form

$$-S = \int_C (m\sigma_{\mathbf{a}} + qA_{\mathbf{a}}) dx^{\mathbf{a}} \quad (2.3.9)$$

$$= \int_C \left( m\sqrt{g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}} + qA_{\mathbf{a}}\dot{x}^{\mathbf{a}} \right) dt \quad (2.3.10)$$

The Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} \right) = \frac{\partial L}{\partial x^{\mathbf{a}}} \quad (2.3.11)$$

gives

$$\frac{d}{dt} (p_{\mathbf{a}} + qA_{\mathbf{a}}) = q(\nabla_{\mathbf{a}}A_{\mathbf{b}}) \frac{dx^{\mathbf{b}}}{dt} \quad (2.3.12)$$

where the particle's momentum <sup>1</sup>

$$p_{\mathbf{a}} = \frac{mg_{\mathbf{ab}}}{\sqrt{g_{\mathbf{cd}}\dot{x}^{\mathbf{c}}\dot{x}^{\mathbf{d}}}} \frac{dx^{\mathbf{b}}}{dt} = mg_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{d\tau} \quad (2.3.13)$$

Therefore the force on the particle is

$$f_{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{d\tau} = mg_{\mathbf{ab}} \frac{d^2x^{\mathbf{b}}}{d\tau^2} = qF_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{d\tau} \quad (2.3.14)$$

where the electromagnetic field

$$F_{\mathbf{ab}} = \nabla_{\mathbf{a}}A_{\mathbf{b}} - \nabla_{\mathbf{b}}A_{\mathbf{a}} \quad (2.3.15)$$

### 2.3.3 Newtonian decomposition

Following Eq. (2.2.7), the spacetime velocity decomposes as

$$u^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{d\tau} = \frac{dt}{d\tau} (e_t^{\mathbf{a}} + v^{\mathbf{a}}) \quad (2.3.16)$$

where the spatial velocity

$$v^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{dt} \quad (2.3.17)$$

the spacetime force decomposes as

$$f_{\mathbf{a}} = \frac{dt}{d\tau} (Pe_{\mathbf{a}}^t - F_{\mathbf{a}}) \quad (2.3.18)$$

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<sup>1</sup>Note that  $p_{\mathbf{a}} = m\sigma_{\mathbf{a}}$ .

where the spatial force satisfies

$$e_t^{\mathbf{a}} F_{\mathbf{a}} = 0 \quad (2.3.19)$$

and the electromagnetic field decomposes as

$$F_{\mathbf{ab}} = e_{\mathbf{a}}^t E_{\mathbf{b}} - E_{\mathbf{a}} e_{\mathbf{b}}^t - B_{\mathbf{ab}} \quad (2.3.20)$$

where the electric field  $E_{\mathbf{a}}$  and magnetic flux density  $B_{\mathbf{ab}}$  satisfy

$$e_t^{\mathbf{a}} E_{\mathbf{a}} = 0 \quad (2.3.21)$$

$$e_t^{\mathbf{a}} B_{\mathbf{ab}} = 0 \quad (2.3.22)$$

Substituting Eqs. (2.3.16), (2.3.18) and (2.3.20) into Eq. (2.3.14) gives

$$P e_{\mathbf{a}}^t - F_{\mathbf{a}} = q (e_{\mathbf{a}}^t E_{\mathbf{b}} - E_{\mathbf{a}} e_{\mathbf{b}}^t - B_{\mathbf{ab}}) (e_t^{\mathbf{b}} + v^{\mathbf{b}}) \quad (2.3.23)$$

$$= q (e_{\mathbf{a}}^t E_{\mathbf{b}} v^{\mathbf{b}} - E_{\mathbf{a}} - B_{\mathbf{ab}} v^{\mathbf{b}}) \quad (2.3.24)$$

Therefore Eq. (2.3.14) decomposes into the electromagnetic power equation

$$P = q E_{\mathbf{a}} v^{\mathbf{a}} \quad (2.3.25)$$

and the Lorentz force law

$$F_{\mathbf{a}} = q (E_{\mathbf{a}} + B_{\mathbf{ab}} v^{\mathbf{b}}) \quad (2.3.26)$$