2.4 Dynamics of spacetime

2.4.1 Einstein equation

A spacetime M should have action of the form

$$-S[g_{\mathbf{ab}}] = \int_M \mathcal{L}(\nabla_{\mathbf{a}}, g_{\mathbf{ab}}, R_{\mathbf{abcd}})\,\boldsymbol{\epsilon} - S_{\text{matter}} \tag{2.4.1}$$

where $\mathcal{L}$ is a scalar function and $\boldsymbol{\epsilon}$ is the spacetime volume form. Using dimensional analysis, this can be expanded in a Taylor series in $\hbar$, and in the general relativistic limit of $\hbar \to 0$, see Figure 2.1.1, the higher order terms in $\hbar$ can be neglected leaving

$$-S[g_{\mathbf{ab}}] = \int_M \left[\Lambda + \frac{1}{2}\left(\frac{R}{8\pi G}\right) + O\big(\hbar R^2\big)\right]\boldsymbol{\epsilon} - S_{\text{matter}} \tag{2.4.2}$$

where Λ is the **cosmological constant**, R is the **Ricci scalar**, see Eq. (A1.1.9), and the **gravitational constant** G is defined by this action.

Dimensional analysis would suggest

$$\Lambda \sim \frac{c^7}{\hbar(8\pi G)^2} \tag{2.4.3}$$

which would destabilise the above expansion in $\hbar$, and is over one hundred orders of magnitude larger than the observed value. We will assume that Λ is small, and return to the question of why later.

Using

$$\frac{\partial R}{\partial g^{\mathbf{ab}}} = \frac{\partial}{\partial g^{\mathbf{ab}}}\left(g^{\mathbf{cd}} R_{\mathbf{ced}}{}^{\mathbf{e}}\right) = R_{\mathbf{aeb}}{}^{\mathbf{e}} = R_{\mathbf{ab}} \tag{2.4.4}$$

and

$$\frac{\partial \epsilon_{\mathbf{c_1}\cdots\mathbf{c_n}}}{\partial g^{\mathbf{ab}}} = -\frac{1}{2}\epsilon_{\mathbf{c_1}\cdots\mathbf{c_n}} g_{\mathbf{ab}} \tag{2.4.5}$$

gives

$$-\frac{\delta S}{\delta g^{\mathbf{ab}}} = -\frac{1}{2}\Lambda g_{\mathbf{ab}} + \frac{1}{2}\left(\frac{R_{\mathbf{ab}}}{8\pi G}\right) - \frac{1}{4}\left(\frac{R g_{\mathbf{ab}}}{8\pi G}\right) - \frac{\delta S_{\text{matter}}}{\delta g^{\mathbf{ab}}} \tag{2.4.6}$$

Defining the **Einstein tensor**

$$G_{\mathbf{ab}} \equiv R_{\mathbf{ab}} - \frac{1}{2} R g_{\mathbf{ab}} \tag{2.4.7}$$

the **stress tensor**

$$T_{\mathbf{ab}} \equiv 2\frac{\delta S_{\text{matter}}}{\delta g^{\mathbf{ab}}} \tag{2.4.8}$$

and setting $\delta S = 0$ gives the **Einstein equation**

$$G_{\mathbf{ab}} = 8\pi G\left(\Lambda g_{\mathbf{ab}} + T_{\mathbf{ab}}\right) \tag{2.4.9}$$

Henceforth, we will usually set $8\pi G = 1$.

2.4.2 Matter

A **scalar field** has action

$$-S[\phi] = \int \left[V(\phi) - \frac{1}{2} g^{\mathbf{ab}} \nabla_{\mathbf{a}} \phi \nabla_{\mathbf{b}} \phi \right] \boldsymbol{\epsilon} \tag{2.4.10}$$

equation of motion

$$-\frac{\delta S}{\delta \phi} = \frac{\partial V}{\partial \phi} + g^{\mathbf{ab}} \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} \phi = 0 \tag{2.4.11}$$

and Eqs. (2.4.8) and (2.4.5) give its stress tensor

$$T_{\mathbf{ab}} = V(\phi)\, g_{\mathbf{ab}} + \nabla_{\mathbf{a}} \phi \nabla_{\mathbf{b}} \phi - \frac{1}{2} \left(g^{\mathbf{cd}} \nabla_{\mathbf{c}} \phi \nabla_{\mathbf{d}} \phi \right) g_{\mathbf{ab}} \tag{2.4.12}$$

A **perfect fluid** has stress tensor

$$T_{\mathbf{ab}} = \rho u_{\mathbf{a}} u_{\mathbf{b}} + p \left(u_{\mathbf{a}} u_{\mathbf{b}} - g_{\mathbf{ab}} \right) \tag{2.4.13}$$

where $u^{\mathbf{a}}$ is the fluid velocity and

$$\rho = u^{\mathbf{a}} u^{\mathbf{b}} T_{\mathbf{ab}} \tag{2.4.14}$$

and

$$p = \frac{1}{3} \left(u^{\mathbf{a}} u^{\mathbf{b}} - g^{\mathbf{ab}} \right) T_{\mathbf{ab}} \tag{2.4.15}$$

are the fluid energy density and pressure in the rest frame of the fluid. A perfect fluid gives a good effective description of radiation ($p = \rho/3$) or non-relativistic matter ($p \simeq 0$).

Comparing Eqs. (2.4.9) and (2.4.13), we see that the cosmological constant behaves like a perfect fluid with $\rho = -p = \Lambda$, which can be interpreted as vacuum energy density since $dE = -p\, dV = \rho\, dV$, see Figure 2.4.1.

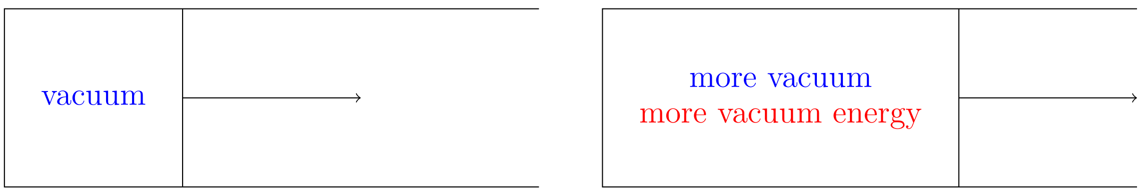

Figure 2.4.1: What happens when you expand a vacuum?

Comparing Eqs. (2.4.2) and (2.4.10) or Eqs. (2.4.9) and (2.4.12), we see that the potential of a scalar field acts like a field dependent cosmological "constant". See Figure 2.4.2.

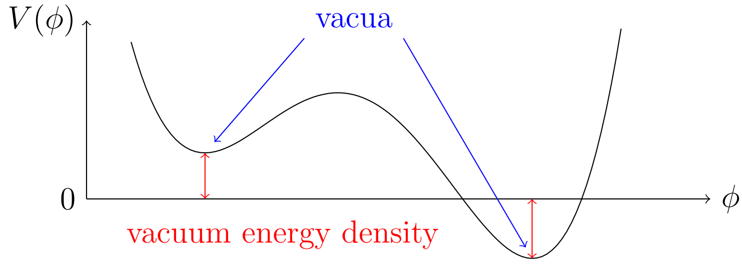

Figure 2.4.2: Cosmological "constant".