2.7 Gravitational waves

2.7.1 Weyl tensor

The curvature tensor can be decomposed in terms of the **Weyl tensor** and the Ricci tensor and scalar

$$\begin{aligned} R_{\mathbf{abcd}} &= C_{\mathbf{abcd}} + \frac{1}{N-2}\left(R_{\mathbf{ac}}g_{\mathbf{bd}} - R_{\mathbf{ad}}g_{\mathbf{bc}} - R_{\mathbf{bc}}g_{\mathbf{ad}} + R_{\mathbf{bd}}g_{\mathbf{ac}}\right) \\ &\quad - \frac{1}{(N-1)(N-2)}R\left(g_{\mathbf{ac}}g_{\mathbf{bd}} - g_{\mathbf{ad}}g_{\mathbf{bc}}\right) \end{aligned} \tag{2.7.1}$$

where N is the dimension of the space. The Ricci tensor and scalar are determined by the matter via the Einstein equation. The Weyl tensor represents gravitational waves.

2.7.2 Plane waves

In Rosen coordinates, the metric for a gravitational plane wave is

$$d\tau^2 = 2\,du\,dv - \gamma_{\kappa\lambda}(u)\,dx^\kappa\,dx^\lambda \tag{2.7.2}$$

where u and v are null coordinates

$$u = \frac{1}{\sqrt{2}}(t-z) \tag{2.7.3}$$

$$v = \frac{1}{\sqrt{2}}(t+z) \tag{2.7.4}$$

and the $x^\kappa = x, y$ are transverse coordinates. The non-zero Christoffel symbols are

$$\Gamma^v_{\kappa\lambda} = \frac{1}{2}g^{vu}\left(-g_{\kappa\lambda,u}\right) = \frac{1}{2}\gamma'_{\kappa\lambda} \tag{2.7.5}$$

$$\Gamma^\kappa_{u\lambda} = \frac{1}{2}g^{\kappa\mu}\left(g_{\mu\lambda,u}\right) = \frac{1}{2}\gamma^{\kappa\mu}\gamma'_{\mu\lambda} \tag{2.7.6}$$

plus those related by symmetry. The geodesic equation gives

$$\frac{du}{d\tau} = \text{constant} \tag{2.7.7}$$

$$\frac{d^2v}{d\tau^2} + \frac{1}{2}\gamma'_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{2.7.8}$$

and

$$\gamma_{\kappa\lambda}\frac{dx^\lambda}{d\tau} = \text{constant} \tag{2.7.9}$$

and so worldlines with $x^\kappa = \text{constant}$ and $dz/dt = \text{constant}$ are geodesics despite the fact that the transverse distance between them is varying according to $\gamma_{\kappa\lambda}(u)$. The non-zero component of the Einstein tensor is

$$G_{uu} = -\frac{1}{2}\gamma^{\kappa\lambda}\gamma''_{\lambda\kappa} + \frac{1}{4}\gamma^{\kappa\lambda}\gamma'_{\lambda\mu}\gamma^{\mu\nu}\gamma'_{\nu\kappa} \tag{2.7.10}$$

and so the vacuum Einstein equation reduces to

$$\gamma^{\kappa\lambda}\gamma''_{\lambda\kappa} - \frac{1}{2}\gamma^{\kappa\lambda}\gamma'_{\lambda\mu}\gamma^{\mu\nu}\gamma'_{\nu\kappa} = 0 \tag{2.7.11}$$

For small amplitude gravitational waves, we can write

$$\gamma_{\kappa\lambda}(u) = \delta_{\kappa\lambda} + h_{\kappa\lambda}(u) \tag{2.7.12}$$

with $h_{\kappa\lambda}$ small. Then the Einstein equation linearizes to

$$h''_{xx} + h''_{yy} + O\big(h^2\big) = 0 \tag{2.7.13}$$

showing the traceless nature of a gravitational wave.

In Brinkmann coordinates, the metric for a gravitational plane wave is

$$d\tau^2 = 2\,du\,dv + f_{\kappa\lambda}(u)\,x^\kappa x^\lambda\,du^2 - \delta_{\kappa\lambda}\,dx^\kappa\,dx^\lambda \tag{2.7.14}$$

The non-zero Christoffel symbols are

$$\Gamma^\kappa_{uu} = \frac{1}{2}g^{\kappa\mu}\left(-g_{uu,\mu}\right) = \delta^{\kappa\mu}f_{\mu\lambda}x^\lambda \tag{2.7.15}$$

$$\Gamma^v_{u\kappa} = \frac{1}{2}g^{vu}\left(g_{uu,\kappa}\right) = f_{\kappa\lambda}x^\lambda \tag{2.7.16}$$

$$\Gamma^v_{uu} = \frac{1}{2}g^{vu}\left(g_{uu,u}\right) = \frac{1}{2}f'_{\kappa\lambda}x^\kappa x^\lambda \tag{2.7.17}$$

The geodesic equation gives

$$\frac{du}{d\tau} = \text{constant} \tag{2.7.18}$$

$$\frac{d^2v}{d\tau^2} + 2f_{\kappa\lambda}x^\lambda\frac{dx^\kappa}{d\tau}\frac{du}{d\tau} + \frac{1}{2}f'_{\kappa\lambda}x^\kappa x^\lambda\left(\frac{du}{d\tau}\right)^2 = 0 \tag{2.7.19}$$

and

$$\frac{d^2x^\kappa}{d\tau^2} + \delta^{\kappa\mu}f_{\mu\lambda}x^\lambda\left(\frac{du}{d\tau}\right)^2 = 0 \tag{2.7.20}$$

and so worldlines with $x^\kappa = 0$ and $dz/dt =$ constant are geodesics and neighboring geodesics undergo accelerated transverse motion governed by $f_{\kappa\lambda}(u)$. The non-zero component of the Einstein tensor is

$$G_{uu} = \delta^{\kappa\lambda}f_{\lambda\kappa} \tag{2.7.21}$$

and so the vacuum Einstein equation reduces to

$$f_{xx} + f_{yy} = 0 \tag{2.7.22}$$

again showing the traceless nature of a gravitational wave.