3 Inflation

3.1 Motivation

In this section we will try to understand some of the basic observed properties of the universe listed in Section 1.2. For simplicity we will assume approximate homogeneity and isotropy.

5. & 1. **Expanding** and **old**
General relativity tells us that the universe is dynamical and so would be expected to be either expanding or contracting. From

$$3H^2 + 3K = \rho \tag{121}$$

we see that if $\rho \geq 0$ and $K \leq 0$ the universe will expand forever.

2. **Big**
An old expanding universe should be big, but how big? It would be natural to create a Planck size universe expanding at the Planck rate, $L \sim H \sim 1$. The current value of the Hubble parameter is $H_0 \sim 10^{-60}$, and so for a universe dominated by radiation or matter since the Planck epoch the current size of the universe would be $L_0 \sim 10^{30} \sim 0.1\,\text{mm}$ or $L_0 \sim 10^{40} \sim 10^3\,\text{km}$ respectively. We know the universe is bigger: $L_0 \gtrsim 1/H_0 \sim 10^{60}$. To start with $LH \sim 1$ and end up with $LH \gtrsim 1$ we need

$$\frac{d}{dt}(LH) \geq 0\,, \quad \text{i.e.} \quad \ddot{a} \geq 0 \tag{122}$$

3. **A lot of matter**
The universe contains a lot of matter, $M_0 \gtrsim 10^{60}$. Where did it all come from? To create matter or energy in an expanding universe requires $p < 0$. [1] However, we don't just need to create energy, we need to create energy rapidly enough to get $E_0 \gtrsim \rho_0 (1/H_0)^3 \sim \rho_0^{-1/2}$. It is natural to expect the universe to be created with a Planck mass of energy, $E \sim 1$, at the Planck density, $\rho \sim 1$. To go from $E \sim \rho^{-1/2}$ to $E \gtrsim \rho^{-1/2}$ requires

$$\frac{d\ln E}{d\ln a} \geq -\frac{1}{2}\frac{d\ln\rho}{d\ln a}\,, \quad \text{i.e.} \quad \frac{d\ln E}{d\ln a} \geq 1 \tag{123}$$

This requires $p \leq -\rho/3$ which gives $\ddot{a} \geq 0$.

6. **No observable spatial curvature**
We know that the spatial curvature is now smaller than the energy density, $|K_0| < \rho_0 \sim 10^{-120}$. The initial conditions at the Planck epoch ($\rho \sim 1$) required to achieve this in a universe dominated by radiation or matter are $|K| \lesssim 10^{-60}$ or $|K| \lesssim 10^{-40}$ respectively. Even at the time of nucleosynthesis it requires $|K| \lesssim 10^{-15}\rho$. These can hardly be regarded as sensible initial conditions. Instead it would be natural to create a universe with $|K| \sim \rho \sim 1$. To evolve from $|K| \sim \rho$ to $|K| < \rho$ requires

$$-\frac{d\ln\rho}{d\ln a} \leq -\frac{d\ln K}{d\ln a} = 2\,, \quad \text{i.e.} \quad \ddot{a} \geq 0 \tag{124}$$

[1] Negative gravitational potential energy is generated at the same time so the total energy is conserved.

Thus, the fact that the universe is big, $L_0 \gtrsim 1/H_0$, that it contains a lot of matter, $M_0 \gtrsim 1/H_0$, and that it has no observable spatial curvature, $|K_0| < H_0^2$, all suggest that

$$\ddot{a} \geq 0 \tag{125}$$

on average during the history of the universe. We know that much of the recent expansion of the universe occurred with $\ddot{a} < 0$, and so require a sufficiently long earlier epoch with $\ddot{a} > 0$.

4. **Homogeneous** and **isotropic**
   We are assuming approximate homogeneity and isotropy and so all we can hope to explain is how to make an approximately homogeneous and isotropic universe more homogeneous and isotropic on the appropriate scales.

   If $\ddot{a} > 0$, comoving scales leave the horizon. Therefore comoving inhomogeneities and anisotropies will get stretched beyond the horizon. $\ddot{a} > 0$ also implies that ρ decreases more slowly than a^{-2}. Therefore discrete inhomogeneities will get diluted. Thus if $\ddot{a} > 0$, scales fixed relative to the horizon will tend to become homogeneous and isotropic.

   Note that any unwanted relics (particles, black holes, topological defects) from the early universe can be viewed as inhomogeneities and are got rid of in the same way.

7. **Density perturbations**
   If $\ddot{a} > 0$, vacuum (or even thermal) fluctuations on subhorizon scales can be magnified into classical perturbations on superhorizon scales. This will be the subject of Section 3.6.

References

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3. A. D. Linde, Physics Letters B108 (1982) 389-393; A. Albrecht and P. J. Steinhardt, Physical Review Letters 48 (1982) 1220-1223.

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3.2 Definition of inflation

We have seen that many of the basic observed properties of the universe can be explained by a sufficiently long epoch with $\ddot{a} > 0$. This is the usual definition of inflation.

Inflation is characterized by

Repulsive gravity

$$\ddot{a} > 0 \tag{126}$$

Comoving scales leave the horizon

$$\frac{d}{dt}\left(\frac{\lambda_{\mathrm{phys}}}{1/H}\right) > 0 \tag{127}$$

Sufficiently slowly decreasing Hubble parameter

$$-\frac{d\ln H}{d\ln a} < 1 \tag{128}$$

Curvature decreases relative to the energy density

$$-\frac{d\ln K}{d\ln a} > -\frac{d\ln\rho}{d\ln a} \tag{129}$$

Sufficiently negative pressure

$$p < -\frac{1}{3}\rho \tag{130}$$

The amount of inflation is measured by

$$\mathcal{N} = \ln\dot{a} = \ln(aH) \tag{131}$$

During inflation, H is usually approximately constant, and so $\mathcal{N}$ is approximately equivalent to the number of e-folds of expansion

$$N = \ln a = \int H dt \tag{132}$$

which is what is usually used to describe the time during inflation.

It is useful to know when a given comoving scale k crosses the horizon during inflation. Defining horizon crossing to occur when $aH = k$, a comoving scale crosses the horizon a number of e-folds N before the end of inflation given by

$$N = N_{\mathrm{end}} - N_{\mathrm{cross}} = \mathcal{N}_{\mathrm{end}} - \mathcal{N}_{\mathrm{cross}} + \ln\left(\frac{H_{\mathrm{cross}}}{H_{\mathrm{end}}}\right) \tag{133}$$

and

$$\begin{aligned}\mathcal{N}_{\mathrm{end}} - \mathcal{N}_{\mathrm{cross}} &= (\mathcal{N}_{\mathrm{end}} - \mathcal{N}_{\mathrm{nuc}}) + (\mathcal{N}_{\mathrm{nuc}} - \mathcal{N}_0) - (\mathcal{N}_{\mathrm{cross}} - \mathcal{N}_0) \\ &= (\mathcal{N}_{\mathrm{end}} - \mathcal{N}_{\mathrm{nuc}}) + 20 - \ln\left(\frac{k}{a_0 H_0}\right) \qquad (134)\end{aligned}$$

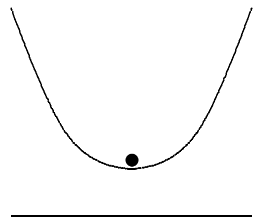
Subscript nuc denotes the beginning of nucleosynthesis, which is taken to be when the temperature was $T = 10\,\mathrm{MeV}$. This is the earliest time at which the evolution of the universe is well understood and observationally tested. What happened before nucleosynthesis is highly speculative. $\mathcal{N}_{\mathrm{end}} - \mathcal{N}_{\mathrm{nuc}}$ could be positive or negative, with positive values restricted to be $\lesssim 40$. The current scales of observational cosmology span the range $\ln(k/a_0 H_0) \sim 0$ to 15.

3.3 Types of inflation

Inflation requires a form of matter with $p < -\frac{1}{3}\rho$. This can be provided by positive vacuum energy density, or, more generally, by the potential energy density of a scalar field, both of which have $p = -\rho$.

In this section we will describe the types of inflation that emerge naturally from particle physics, and also the type that is required to produce an approximately scale-invariant spectrum of density perturbations.

3.3.1 Positive cosmological constant



This is the simplest type of inflation. The energy density of the universe is dominated by a positive cosmological constant, i.e. the positive energy density of our vacuum. The universe then expands exponentially

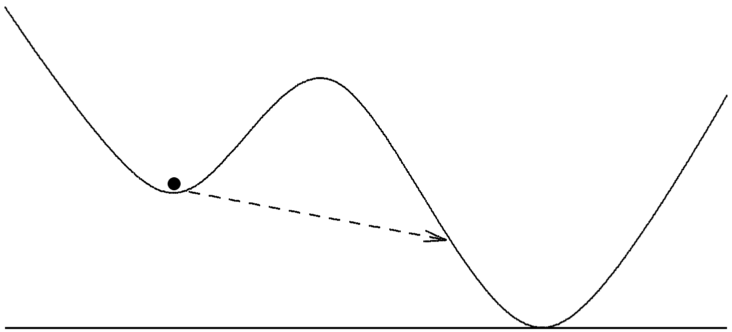
$$a \propto e^{Ht}, \quad H = \text{constant} \tag{135}$$

The energy density in any radiation or matter decays exponentially, as does the curvature

$$\rho_{\text{rad}} \propto e^{-4Ht}, \quad \rho_{\text{mat}} \propto e^{-3Ht}, \quad K \propto e^{-2Ht} \tag{136}$$

This type of inflation **never ends** and so cannot be the origin of our hot Big Bang universe.

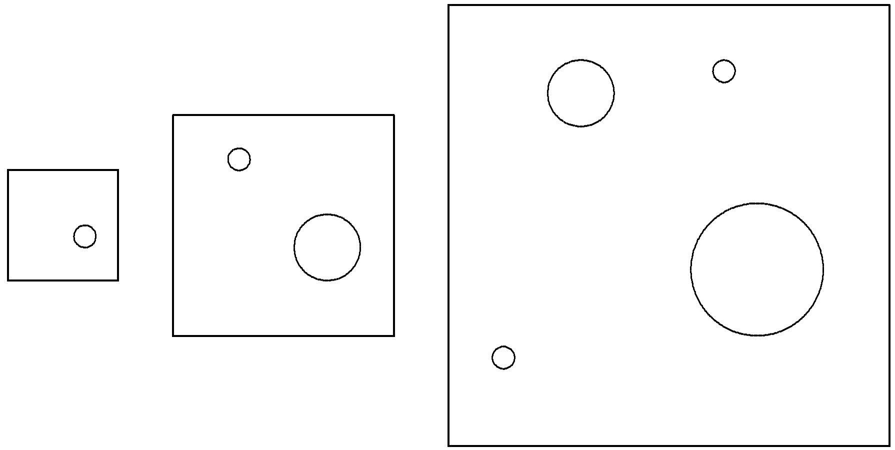
3.3.2 False vacuum inflation



Here the positive vacuum energy is that of a false vacuum so that inflation can end by quantum tunneling to the true vacuum. If the decay rate per unit volume $\Gamma \gtrsim H^4$, inflation will barely begin, so we will assume $\Gamma \lesssim H^4$.

Once the universe becomes trapped in the false vacuum, spacetime will tend to de Sitter space, which contains the exponentially large, flat, homogeneous and isotropic, spatial hypersurfaces that we want inflation to produce. However, because de Sitter space is not just spatially homogeneous, but is completely homogeneous, it has **no clock**, i.e. no unique choice of time-slicing, to distinguish these hypersurfaces from any others. The quantum tunneling is a random Poisson process and so also does not have a clock. Therefore, the end of inflation cannot be synchronized to occur on one of these exponentially large, flat, homogeneous and isotropic, spatial hypersurfaces, and so the achievements of the inflation are not preserved by the exit from inflation.

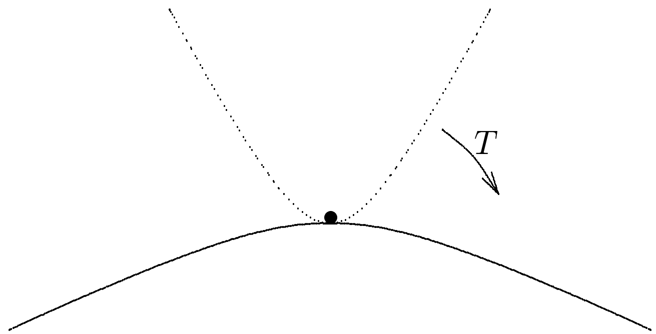
Indeed, if $\Gamma \lesssim H^4$, the inflation never ends completely because the volume of the universe which is inflating increases at a rate faster than it can be eaten up by the nucleating and expanding bubbles of true vacuum. One gets an **eternally inflating universe** continually nucleating bubbles of true vacuum. Each of these bubbles of true vacuum corresponds to



an infinite, negative curvature dominated, homogeneous and isotropic universe. Thus if a subsequent more phenomenologically acceptable inflation occurred within some of these bubble universes, one could have an eternally inflating universe filled with an infinite number of hot Big Bang bubble universes.

3.3.3 Thermal inflation

Here the finite temperature effective potential provides the false vacuum, and the temperature acts as a clock to synchronize the end of inflation.



Consider the finite temperature effective potential

$$V = V_0 + \frac{1}{2}\left(g^2T^2 - m^2\right)\phi^2 + \ldots \tag{137}$$

with $g \sim 1$. When

$$m \lesssim T \lesssim V_0^{1/4} \tag{138}$$

the scalar field ϕ is held at $\phi = 0$ by the finite temperature T, and the false vacuum energy density V_0 dominates the thermal energy density $\sim T^4$ and so drives an epoch of inflation.

The temperature T decreases during the inflation as

$$T \propto \frac{1}{a} \propto e^{-Ht}, \quad H = \sqrt{\frac{V_0}{3}} \tag{139}$$

and acts as a clock to synchronize the end of inflation. The inflation lasts for

$$N \sim \ln\left(\frac{T_{\text{initial}}}{T_{\text{final}}}\right) \sim \ln\left(\frac{V_0^{1/4}}{m}\right) \tag{140}$$

e-folds. To get a significant amount of inflation we require

$$m \ll V_0^{1/4} \tag{141}$$

For example, potentials of the form

$$V = V_0 - \frac{1}{2}m^2\phi^2 + \frac{\lambda^2}{M_{\text{Pl}}^{2n}}\phi^{4+2n}, \quad n \geq 1 \tag{142}$$

which are common in supersymmetric theories, have their minimum at $\phi \equiv M \gg m$, and so in order to cancel the cosmological constant have $V_0 \sim m^2M^2 \gg m^4$. For $m \sim m_{\text{EW}} \sim 10^{-16}$ and $\lambda \sim 1$ we get

$$N \sim \frac{n}{2(n+1)}\ln\left(\frac{1}{m}\right) \sim 18\left(\frac{n}{n+1}\right) \tag{143}$$

Thus, even for $n = \infty$, which corresponds to $M = M_{\text{Pl}}$, a single epoch of thermal inflation does **not** give **enough *e*-folds** of inflation to make the universe big, flat, etc.

Also, thermal inflation is **not** a **scale-invariant** process, the temperature falls like $T \propto 1/a$, and so can not produce a scale-invariant spectrum of density perturbations.

3.3.4 Rolling scalar field inflation

Here a rolling scalar field provides the clock that synchronizes the end of inflation.

Inflation requires $p < -\rho/3$, and for a homogeneous scalar field

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \tag{144}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V \tag{145}$$

and so to get inflation we need

$$\dot{\phi}^2 < V \tag{146}$$

This is most simply achieved near a maximum of the potential

$$V = V_0 - \frac{1}{2}m^2\phi^2 + \ldots \tag{147}$$

The equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{148}$$

The friction term $3H\dot{\phi}$ arises from the expansion of the universe which damps the kinetic energy.

If $\phi = 0$, or is sufficiently small, the dynamics will be dominated by quantum fluctuations. Using the results of Section 3.4, one can show that one gets **eternal inflation** in the neighborhood of $\phi = 0$ if $m^2 \leq 6V_0$.

Once ϕ escapes from the neighborhood of zero, which we crudely take to occur when $\phi \gtrsim H$, the classical motion will dominate. While ϕ is still near the top of the potential, $\phi \ll V_0^{1/2}/m$, we have $H \simeq \sqrt{V_0/3}$, and so can solve the equation of motion to give

$$\phi \propto a^\alpha \tag{149}$$

where

$$\alpha = \frac{3}{2}\left(\sqrt{1 + \frac{4m^2}{3V_0}} - 1\right) \tag{150}$$

Inflation will end when $\phi \sim V_0^{1/2}/m$. Therefore the number of e-folds of this classical rolling inflation is

$$N \sim \frac{1}{\alpha}\ln\left(\frac{\phi_{\text{end}}}{\phi_{\text{initial}}}\right) \sim \frac{1}{\alpha}\ln\left(\frac{1}{m}\right) \tag{151}$$

For $m \sim m_{\text{EW}} \sim 10^{-16}$ this gives

$$N \sim \frac{37}{\alpha} \tag{152}$$

To obtain a significant amount of inflation we require

$$m \lesssim \frac{V_0^{1/2}}{M_{\text{Pl}}} \tag{153}$$

which is a much stronger constraint than that required by thermal inflation ($m \ll V_0^{1/4}$).

From Eq. (149) we see that rolling scalar field inflation is **not scale-invariant**, unless $\alpha \ll 1$ which is the slow-roll limit to be discussed the next section, and so will not in general produce a scale-invariant spectrum of density perturbations.

3.4 Quantum fields in de Sitter space

In this section we will investigate the behavior of a quantized scalar field in de Sitter space.

For simplicity, we will assume that the spacetime is homogeneous and isotropic despite the fact that the scalar field is not, i.e. we will neglect the back-reaction of the fluctuations in the scalar field on the metric. This will be consistent if the perturbations in the energy density, pressure, etc., are negligible. For example, the behavior of the scalar field near the maximum of the potential in rolling scalar field inflation (Section 3.3.4) can be described using this formalism.

In a homogeneous and isotropic expanding universe with metric

$$ds^2 = dt^2 - a(t)^2 \mathbf{dx}^2 \tag{162}$$

the action for a free massive real scalar field

$$S = \int \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - m^2 \phi^2 \right] \sqrt{-g} \, d^4 x \tag{163}$$

becomes

$$S = \int \frac{1}{2} \left[\dot{\phi}^2 - \frac{1}{a^2} \left(\boldsymbol{\nabla} \phi \right)^2 - m^2 \phi^2 \right] a^3 \, dt \, d^3\mathbf{x} \tag{164}$$

Introducing the conformal time η

$$d\eta = \frac{dt}{a} \tag{165}$$

which is defined to make the metric conformally flat

$$ds^2 = a(\eta)^2 \left[d\eta^2 - \mathbf{dx}^2 \right] \tag{166}$$

denoting the derivative with respect to η by a prime

$$\phi' = a\dot{\phi} \tag{167}$$

and defining

$$\varphi = a\phi \tag{168}$$

gives

$$S = \int \frac{1}{2} \left[\varphi'^2 - \left(\boldsymbol{\nabla} \varphi \right)^2 - \left(a^2 m^2 - \frac{a''}{a} \right) \varphi^2 - \left(\frac{a'}{a} \varphi^2 \right)' \right] d\eta \, d^3\mathbf{x} \tag{169}$$

The equation of motion is

$$\varphi'' - \nabla^2 \varphi + \left(a^2 m^2 - \frac{a''}{a} \right) \varphi = 0 \tag{170}$$

This has the general solution

$$\varphi(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_\mathbf{k} \, \varphi_k(\eta) + a^\dagger_{-\mathbf{k}} \, \varphi^*_k(\eta) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{171}$$

where φ_k satisfies

$$\varphi''_k + \left(k^2 + a^2 m^2 - \frac{a''}{a} \right) \varphi_k = 0 \tag{172}$$

and is normalized such that

$$\varphi_k \varphi^{*\prime}_k - \varphi'_k \varphi^*_k = i \tag{173}$$

The quantization condition

$$\left[\varphi(\eta, \mathbf{x}), \varphi'(\eta, \mathbf{y}) \right] = i \, \delta^3(\mathbf{x} - \mathbf{y}) \tag{174}$$

gives

$$\left[a_\mathbf{k}, a^\dagger_\mathbf{l} \right] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{175}$$

In de Sitter space, H is constant, $a = e^{Ht}$,

$$\eta = -\frac{1}{aH} \tag{176}$$

and Eq. (172) becomes

$$\varphi_k'' + k^2\varphi_k + \frac{1}{\eta^2}\left(\frac{m^2}{H^2} - 2\right)\varphi_k = 0 \tag{177}$$

On scales well inside the horizon, $-k\eta \to \infty$, this reduces to

$$\varphi_k'' + k^2\varphi_k = 0 \tag{178}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(A_k e^{-ik\eta} + B_k e^{ik\eta}\right) , \quad |A_k|^2 - |B_k|^2 = 1 \tag{179}$$

If the inflationary expansion has been going on for sufficiently long, the scalar field should be in the usual flat space vacuum state on scales well inside the horizon. Therefore, we should take $B_k = 0$ so that $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ correspood to the usual flat space annihilation and creation operators, and the state should be $|0\rangle$ where

$$a_{\mathbf{k}}|0\rangle = 0 \tag{180}$$

We are free to take $A_k = 1$ to get

$$\varphi_k \to \frac{1}{\sqrt{2k}} e^{-ik\eta} \qquad \text{as} \quad -k\eta \to \infty \tag{181}$$

The solution of Eq. (177) which matches onto Eq. (181) on scales well inside the horizon is

$$\varphi_k = e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{\frac{\pi}{4k}}\sqrt{-k\eta}\, H_\nu^{(1)}(-k\eta) \tag{182}$$

where

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \tag{183}$$

A useful special case is $m = 0$ which gives $\nu = \frac{3}{2}$ and

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} \tag{184}$$

On scales well outside the horizon we have

$$\varphi_k \to e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} \left(\frac{2^\nu\Gamma(\nu)}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})}\right) \frac{1}{\sqrt{2k}} (-k\eta)^{\frac{1}{2}-\nu} \quad \text{as} \quad -k\eta \to 0 \tag{185}$$

If $m^2 \leq \frac{9}{4}H^2$ then $\nu \in \Re$ and so φ_k and φ_k^* have the same time dependence. This allows us to rewrite the superhorizon Fourier modes, i.e. those with $k \ll aH$, as

$$a_{\mathbf{k}}\,\varphi_k(\eta) + a_{-\mathbf{k}}^\dagger\,\varphi_k^*(\eta) = b_{\mathbf{k}} \left(\frac{2^\nu\Gamma(\nu)}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})}\right) \frac{1}{\sqrt{2k}} (-k\eta)^{\frac{1}{2}-\nu} \tag{186}$$

where

$$b_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} a_{\mathbf{k}} + e^{-i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} a_{-\mathbf{k}}^\dagger \tag{187}$$

Now

$$\left[b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger\right] = 0 \tag{188}$$

and so the superhorizon Fourier modes are **classical** Gaussian random variables with

$$\langle 0|b_{\mathbf{k}} b_{\mathbf{l}}^\dagger|0\rangle = \delta^3(\mathbf{k} - \mathbf{l}) \tag{189}$$

Now

$$\langle 0|\phi^2|0\rangle = \frac{1}{a^2}\langle 0|\varphi^2|0\rangle = \frac{1}{a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,|\varphi_k|^2 \tag{190}$$

Therefore, using Eq. (184), in the special case of $m = 0$

$$\begin{aligned}\langle 0|\phi^2|0\rangle &= \frac{1}{a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left(1+\frac{a^2H^2}{k^2}\right)\\ &= \frac{1}{(2\pi)^2}\int \frac{k\,dk}{a^2} + \left(\frac{H}{2\pi}\right)^2\int\frac{dk}{k}\end{aligned} \tag{191}$$

The first term is the usual short wavelength divergence of quantum field theory and does not concern us here. The second term dominates on the classical superhorizon scales, $k \ll aH$, and gives a long wavelength divergence. We can understand the second term's meaning by restricting the range of integration to be from some fixed physical smoothing scale somewhat larger than the horizon, $k_2/a = \epsilon H$, to some fixed comoving long wavelength cutoff k_1 which crossed the scale k_2/a at $a = a_1$

$$\left(\frac{H}{2\pi}\right)^2 \int_{k_1=\epsilon a_1 H}^{k_2=\epsilon a H}\frac{dk}{k} = \left(\frac{H}{2\pi}\right)^2 \ln\frac{a}{a_1} \tag{192}$$

$\ln(a/a_1)$ is the number of e-folds of expansion N since $k_1 = k_2$. Therefore, the classical superhorizon contribution gives

$$\sqrt{\langle 0|\phi^2|0\rangle} = \frac{H}{2\pi}\sqrt{N} \tag{193}$$

corresponding to a random walk with step length $H/2\pi$ and number of steps N.

Introducing a small mass $m \ll H$ confines the random walk. Then

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2} \tag{194}$$

and, using Eqs. (190) and (185), the classical superhorizon contribution to $\langle 0|\phi^2|0\rangle$ is

$$\begin{aligned}\langle 0|\phi^2|0\rangle &\simeq \frac{1}{a^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left(\frac{aH}{k}\right)^{2\nu-1}\\ &= \left(\frac{H}{2\pi}\right)^2\int\frac{dk}{k}\left(\frac{aH}{k}\right)^{-\frac{2m^2}{3H^2}}\\ &= \frac{3H^4}{8\pi^2m^2}\left[\left(\frac{k}{aH}\right)^{\frac{2m^2}{3H^2}}\right]\end{aligned} \tag{195}$$

Taking the limits of integration to be from $k \sim aH$ to $k = 0$, i.e. all the classical superhorizon scales, gives for $m^2 \ll H^2$

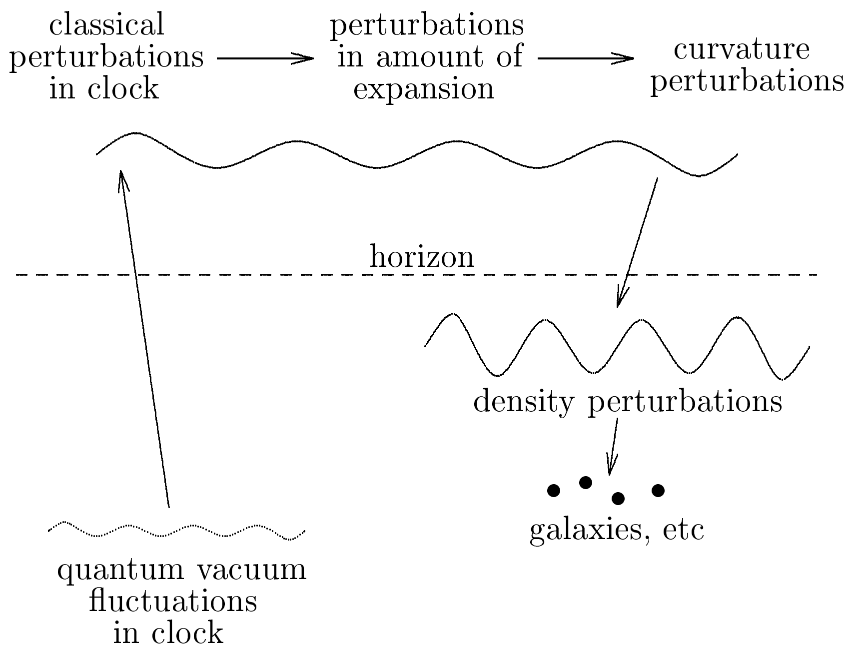
$$\langle 0|\phi^2|0\rangle \simeq \frac{3H^4}{8\pi^2 m^2} \tag{196}$$

References

1. Quantum Fields in Curved Space, N. D. Birrell and P. C. W. Davies, Cambridge University Press (1982, paperback 1984).

3.5 Generating perturbations

The general mechanism for the generation of perturbations during inflation is sketched in the figure. We will first con-



sider the simplest case of slow-roll inflation with a single component inflaton, and then go on to discuss the more general formulation.

3.5.1 Single component inflaton

In Section 3.4 we neglected the perturbations in the metric. Here, we must include them as they are what we will be trying to calculate.

To first order in perturbation theory, the scalar, vector and tensor perturbations decouple from each other. We will focus on the scalar perturbations because they eventually become the density perturbations which grow to form galaxies and the all the rest of the large scale structure in the universe. The tensor perturbations, which correspond to gravitational waves, are in principle also interesting but in practice probably have an unobservably small amplitude. The vector perturbations decay and so are not likely to be interesting.

In the case of a single component inflaton, the action is

$$S = \int \left[-\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right] \sqrt{-g}\, d^4x \qquad (197)$$

There are many different gauge invariant variables we could choose to represent the scalar perturbations. The best choice is

$$\varphi \equiv a \left(\delta\phi - \frac{\dot{\phi}}{H}\mathcal{R} \right) \qquad (198)$$

which is a times the scalar field perturbation on spatially flat hypersurfaces ($-\frac{2}{3}\frac{1}{a^2}\nabla^2\mathcal{R}$ is the spatial curvature perturbation). Once the perturbations leave the horizon, we will want to reinterpret this variable in terms of

$$\mathcal{R}_\mathrm{c} = -\left(\frac{H}{a\dot{\phi}}\right)\varphi = \mathcal{R} - \frac{H}{\dot{\phi}}\delta\phi = \mathcal{R} + H\,(v+B) \qquad (199)$$

which is the curvature perturbation on constant ϕ or comoving ($v + B = 0$) hypersurfaces. $\mathcal{R}_\mathrm{c}$ is a convenient quantity because it is constant on superhorizon scales in a universe containing just a single component of matter (see Section 3.5.2 for clarification and qualification of this statement).

A somewhat lengthy but straightforward calculation gives the action for the scalar perturbations

$$S = \int \frac{1}{2}\left[(\varphi')^2 - (\boldsymbol{\nabla}\varphi)^2 + \frac{H}{a\dot\phi}\left(\frac{a\dot\phi}{H}\right)''\varphi^2\right] d\eta\, d^3\mathbf{x} \tag{200}$$

where a prime denotes the derivative with respect to conformal time η. Note that this includes the metric perturbations coming from both the gravitational and scalar field parts of the action. The equation of motion is

$$\varphi'' - \nabla^2\varphi - \frac{H}{a\dot\phi}\left(\frac{a\dot\phi}{H}\right)''\varphi = 0 \tag{201}$$

This has the general solution

$$\varphi(\eta,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[a_{\mathbf{k}}\,\varphi_k(\eta) + a^\dagger_{-\mathbf{k}}\,\varphi^*_k(\eta)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{202}$$

where φ_k satisfies

$$\varphi_k'' + k^2\varphi_k - \frac{H}{a\dot\phi}\left(\frac{a\dot\phi}{H}\right)''\varphi_k = 0 \tag{203}$$

and is normalized such that

$$\varphi_k\varphi_k^{*\prime} - \varphi_k'\varphi_k^* = i \tag{204}$$

The quantization condition

$$[\varphi(\eta,\mathbf{x}), \varphi'(\eta,\mathbf{y})] = i\,\delta^3(\mathbf{x}-\mathbf{y}) \tag{205}$$

gives

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}}\right] = \delta^3(\mathbf{k}-\mathbf{l}) \tag{206}$$

On small scales we have

$$\varphi_k'' + k^2\varphi_k = 0 \tag{207}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}}e^{-ik\eta} \tag{208}$$

so

$$a_{\mathbf{k}}|0\rangle = 0 \tag{209}$$

corresponds to the usual flat space vacuum on small scales.

On large scales we have

$$\varphi_k'' - \frac{H}{a\dot\phi}\left(\frac{a\dot\phi}{H}\right)''\varphi_k = 0 \tag{210}$$

which has solution

$$\varphi_k = A_k\frac{a\dot\phi}{H} + B_k\frac{a\dot\phi}{H}\int\left(\frac{H}{a\dot\phi}\right)^2 d\eta \tag{211}$$

where A_k and B_k are constants. The growing mode is

$$\varphi_k = A_k\frac{a\dot\phi}{H} \tag{212}$$

Note that φ_k and φ_k^* have the same time dependence.

This allows us to rewrite the large-scale Fourier modes as

$$a_{\mathbf{k}}\,\varphi_k(\eta) + a^\dagger_{-\mathbf{k}}\,\varphi^*_k(\eta) = b_{\mathbf{k}}\frac{a\dot\phi}{H} \tag{213}$$

where

$$b_{\mathbf{k}} = A_k a_{\mathbf{k}} + A^*_k a^\dagger_{-\mathbf{k}} \tag{214}$$

Now

$$\left[b_{\mathbf{k}}, b^\dagger_{\mathbf{l}}\right] = 0 \tag{215}$$

and so the large-scale Fourier modes are **classical** Gaussian random variables with

$$\langle 0|b_{\mathbf{k}}b^\dagger_{\mathbf{l}}|0\rangle = |A_k|^2\,\delta^3(\mathbf{k}-\mathbf{l}) \tag{216}$$

From Eqs. (199) and (213)

$$\mathcal{R}_{\rm c}(\mathbf{k},\eta) = -b_{\mathbf{k}} \tag{217}$$

The $\mathcal{R}_{\rm c}(\mathbf{k},\eta)$ are thus constant, independent, Gaussian magnitude, random phase, classical random variables, and are determined entirely by their **power spectrum**, $P_{\mathcal{R}_{\rm c}}(k)$, which is defined by

$$\langle \mathcal{R}_{\rm c}(\mathbf{k},\eta)\mathcal{R}^*_{\rm c}(\mathbf{l},\eta)\rangle = \frac{2\pi^2}{k^3}P_{\mathcal{R}_{\rm c}}\delta^3(\mathbf{k}-\mathbf{l}) \tag{218}$$

The normalization is chosen so that

$$\langle \mathcal{R}_{\rm c}(\mathbf{x},\eta)\mathcal{R}_{\rm c}(\mathbf{y},\eta)\rangle = \int \frac{dk}{k}P_{\mathcal{R}_{\rm c}}\frac{\sin\left(k\,|\mathbf{x}-\mathbf{y}|\right)}{k\,|\mathbf{x}-\mathbf{y}|} \tag{219}$$

Using Eq. (216), we have

$$P_{\mathcal{R}_{\rm c}}(k) = \frac{k^3}{2\pi^2}\,|A_k|^2 \tag{220}$$

To get our final answer, we need to determine A_k by matching the long wavelength solution, Eq. (212), to the short wavelength solution, Eq. (208). In **slow-roll** inflation, H and $\dot\phi$ are slowly varying, and so we can match the short and long wavelength solutions using an approximate solution which treats H and $\dot\phi$ as constants during horizon crossing. For H and $\dot\phi$ constant, $\eta = -1/(aH)$ and Eq. (203) becomes

$$\varphi''_k + k^2\varphi_k - \frac{2}{\eta^2}\varphi_k = 0 \tag{221}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)e^{-ik\eta} \to \frac{i}{\sqrt{2k}}\frac{aH}{k} \qquad \text{as} \quad \frac{k}{aH}\to 0 \tag{222}$$

Matching this to Eq. (212) gives

$$A_k = \frac{i}{\sqrt{2k^3}}\frac{H^2}{\dot\phi} \tag{223}$$

Within our approximation, we can choose to evaluate the right hand side at any time around horizon crossing. For definiteness, we evaluate it at horizon crossing

$$A_k = \frac{i}{\sqrt{2k^3}}\frac{H^2}{\dot\phi}\bigg|_{aH=k} \tag{224}$$

and so from Eq. (220)

$$P_{\mathcal{R}_{\rm c}}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot\phi}\right)^2\bigg|_{aH=k} \tag{225}$$