Homework 1 - Taylor expansion

A sufficiently smooth function $f(x)$ can be expanded in a **Taylor series** about a point x_0 as

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (x - x_0)^n \tag{H1.1}$$

where

$$a_n = f^{(n)}(x_0) \tag{H1.2}$$

and superscript (n) denotes the nth derivative. In physics, this provides a very useful approximation when $|x - x_0|$ is sufficiently small

$$f(x) = a_0 + a_1 (x - x_0) + \frac{1}{2} a_2 (x - x_0)^2 + \mathcal{O}\big((x - x_0)^3\big) \tag{H1.3}$$

with the first few terms usually being sufficient if the approximation is going to be useful.

Q1.1. By calculating their derivatives, determine the Taylor expansions of e^x, $\cos x$ and $\sin x$ about $x = 0$. Hence show that

$$e^{ix} = \cos x + i \sin x \tag{Q1.1.1}$$

where $i^2 = -1$.

Q1.2. A particle moving vertically above the surface of the Earth experiences a gravitational acceleration

$$\ddot{z} = -\frac{GM_\oplus}{(R_\oplus + z)^2} \tag{Q1.2.1}$$

where z is the height above the surface of the Earth, a dot denotes the time derivative, G is the gravitational constant, and $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth.

What is the gravitational acceleration near the surface of the Earth? Integrate Eq. (Q1.2.1) for motion near the surface of the Earth. How long would it take a particle launched vertically upwards at $100\,\mathrm{m\,s^{-1}}$ to return to the surface of the Earth? Estimate how long it would have to take the particle to return before the errors in your results become large.

Q1.3. Expand the relativistic energy of a massive particle

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{Q1.3.1}$$

in a Taylor series in $v \ll c$ to leading non-trivial order and hence derive the Newtonian energy of a particle. What is the error in the Newtonian formula for $v = 10\,\mathrm{km\,s^{-1}}$? For $v = c/2$?