Homework 2 - Partial differentiation

The derivative $f'(x)$ of a function $f(x)$ can be defined by

$$\delta f \equiv f(x+\delta x) - f(x) = f'(x)\,\delta x + \mathcal{O}\big((\delta x)^2\big) \tag{H2.1}$$

In the limit that δx becomes infinitesimal, $\delta x \to dx$, the higher order terms disappear, leaving

$$df = f'(x)\,dx \tag{H2.2}$$

and hence the notation

$$\frac{df}{dx} \equiv f'(x) \tag{H2.3}$$

For a function $f(x,y)$ of two variables, the infinitesimal change in the function can be written as

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{H2.4}$$

where the **partial derivatives** $\partial f/\partial x$ and $\partial f/\partial y$ are derivatives with respect to x at constant y, and with respect to y at constant x, respectively

$$\frac{\partial f}{\partial x} \equiv \left.\frac{df}{dx}\right|_{dy=0} \tag{H2.5}$$

$$\frac{\partial f}{\partial y} \equiv \left.\frac{df}{dy}\right|_{dx=0} \tag{H2.6}$$

For a function $f(x)$ of position, the infinitesimal change in position is a vector, $\vec{dx}$, and so

$$df = \frac{df}{\vec{dx}} \cdot \vec{dx} \tag{H2.7}$$

with the spatial derivative usually expressed with the notation

$$\underline{\nabla} f \equiv \frac{df}{\vec{dx}} \tag{H2.8}$$

Q2.1. The Lagrangian $L(q,\dot{q},t)$ satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{Q2.1.1}$$

where

$$\dot{q} \equiv \frac{dq}{dt} \tag{Q2.1.2}$$

Show that the Hamiltonian

$$H \equiv p\dot{q} - L \tag{Q2.1.3}$$

where

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{Q2.1.4}$$

is a function of q, p, t and not $\dot{q}$, i.e. $H = H(q, p, t)$, and that

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \tag{Q2.1.5}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{Q2.1.6}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{Q2.1.7}$$

A2.1.

$$H = p\dot{q} - L(q, \dot{q}, t) \tag{A2.1.1}$$

therefore, using Eq. (Q2.1.4),

$$\frac{\partial H}{\partial \dot{q}} = p - \frac{\partial L}{\partial \dot{q}} = 0 \tag{A2.1.2}$$

hence $H = H(q, p, t)$. Also

$$\frac{\partial H}{\partial p} = \dot{q} \tag{A2.1.3}$$

hence Eq. (Q2.1.5) and, using Eq. (Q2.1.1),

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\dot{p} \tag{A2.1.4}$$

hence Eq. (Q2.1.6). Finally, using Eqs. (Q2.1.6) and (Q2.1.5),

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial t} = -\frac{dp}{dt}\frac{dq}{dt} + \frac{dq}{dt}\frac{dp}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{A2.1.5}$$

hence Eq. (Q2.1.7).