Homework 4 - Conserved quantities

Q4.1. A uniform rigid rod of mass $2m$ is in a static, homogeneous and isotropic environment, and is initially at rest. A particle of mass m hits one of its ends elastically and at right angles to the rod.

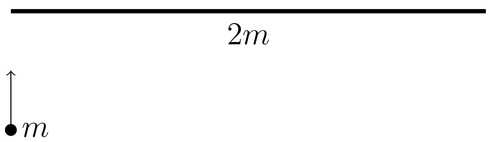

Figure Q4.1.1: Initial state of particle and rod.

Describe the subsequent motion.

A4.1. Due to their static, homogeneous and isotropic environment, the total energy, momentum and angular momentum of the particle and rod will be conserved. When the particle hits the rod it will transfer some energy, momentum and angular momentum to the rod, causing the rod to move forward and rotate. It will be convenient to divide the conserved quantities into the parts due to the motion of: the particle, the center of mass of the rod, and the rod about its center of mass.

Let the initial speed of the particle be u_1 and the state of the particle and rod after the collision be as shown in Figure A4.1.1. Conservation of energy gives

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mu_2^2 + mv_2^2 + \frac{1}{2}I\omega_2^2 \tag{A4.1.1}$$

where the moment of inertia of the rod about its center of mass is

$$I = 2\int_0^l \frac{m}{l} r^2 \, dr = \frac{2}{3}ml^2 \tag{A4.1.2}$$

Conservation of momentum gives

$$mu_1 = mu_2 + 2mv_2 \tag{A4.1.3}$$

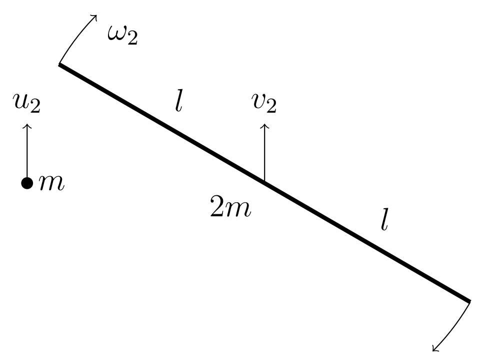

Figure A4.1.1: After the collision.

Conservation of angular momentum about the initial position of the center of mass of the rod gives

$$-mlu_1 = -mlu_2 - I\omega_2 \tag{A4.1.4}$$

Simplifying gives

$$\begin{aligned} u_1^2 &= u_2^2 + 2v_2^2 + \frac{2}{3}l^2\omega_2^2 && \text{(A4.1.5)} \\ u_1 &= u_2 + 2v_2 && \text{(A4.1.6)} \\ u_1 &= u_2 + \frac{2}{3}l\omega_2 && \text{(A4.1.7)} \end{aligned}$$

Therefore

$$l\omega_2 = 3v_2 \tag{A4.1.8}$$

and

$$\begin{aligned} u_1^2 &= u_2^2 + 8v_2^2 && \text{(A4.1.9)} \\ u_1 &= u_2 + 2v_2 && \text{(A4.1.10)} \end{aligned}$$

Therefore

$$u_2 = v_2 = \frac{1}{3}u_1 \tag{A4.1.11}$$

and

$$l\omega_2 = u_1 \tag{A4.1.12}$$

Note that, since $u_2 = v_2$, after a half rotation, i.e. after a time

$$T = \frac{\pi}{\omega_2} = \frac{\pi l}{u_1} \tag{A4.1.13}$$

and hence distance of travel

$$D = \frac{\pi l}{3} \tag{A4.1.14}$$

the rod will collide with the particle again, transfering back energy, momentum and angular momentum, returning the system to the state shown in Figure A4.1.2.

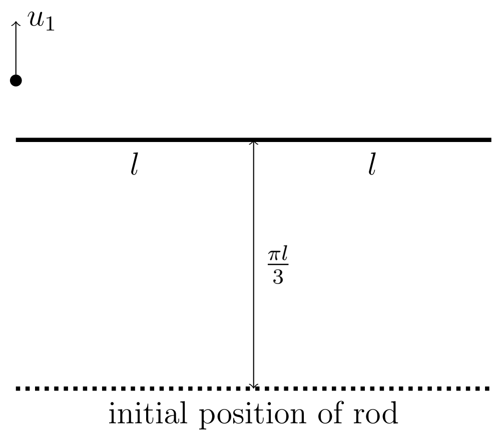

Figure A4.1.2: Final state of particle and rod.