Chapter 2

Classical Mechanics

Mechanics is the branch of physics that deals with the laws of motion.

2.1 Tensors

A **tensor** is a mathematical object that directly represents a physical quantity. Tensors of the same type can be added, and multiplied by a scalar, in the usual way. Scalars and vectors are tensors, but many physical quantities are some other type of tensor.

A **scalar** is a tensor that behaves like a number. Examples of spatial[1] scalars are time t, energy E and electric potential ϕ. Examples of spacetime scalars are proper time τ, mass m and charge q.

2.1.1 Multivectors

A **vector** is a tensor that behaves like an arrow or an oriented line element. Their

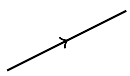

Figure 2.1.1: A vector.

properties inspire the vector space axioms of mathematics. A scalar times a vector is a vector and the sum of two vectors is a vector, see Figure 2.1.2.

Examples of vectors are displacement $\vec{dx}$, velocity

$$\vec{v} \equiv \frac{\vec{dx}}{dt} \tag{2.1.1}$$

and acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \tag{2.1.2}$$

[1]Physical quantities may be one type of tensor with respect to one space but another type of tensor with respect to another space. For example, a displacement in time is a scalar with respect to space but a vector with respect to time. Unless otherwise specified, the space can be assumed to be space, or spacetime in the context of relativity.

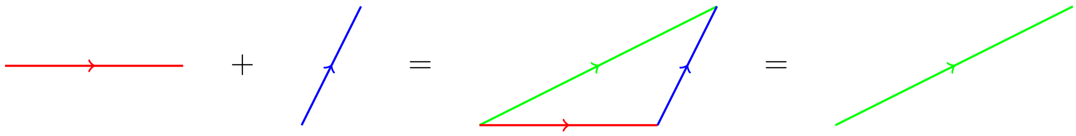

Figure 2.1.2: The sum of two vectors is a vector.

The **exterior** or **wedge product** of two vectors is a **bivector** or **two-vector** given by the oriented plane element formed by the two vectors, see Figure 2.1.3. Note that

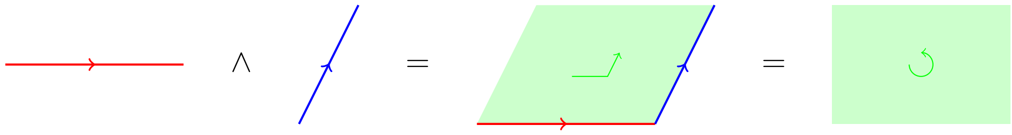

Figure 2.1.3: The exterior product of two vectors is a two-vector.

the shape of the plane element does not matter

$$(2\vec{a}) \wedge \vec{b} = \vec{a} \wedge (2\vec{b}) = 2(\vec{a} \wedge \vec{b}) \tag{2.1.3}$$

only its plane, area and orientation. The exterior product is antisymmetric

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} \tag{2.1.4}$$

since swapping the vectors in Figure 2.1.3 would reverse the orientation, see Figure 2.1.4.

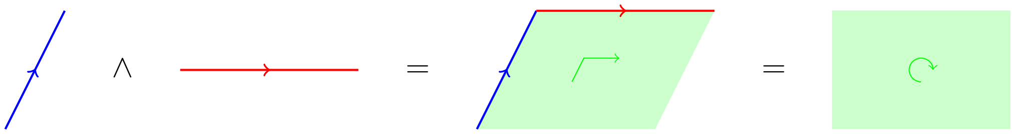

Figure 2.1.4: The exterior product is antisymmetric.

A scalar times a two-vector is a two-vector and the sum of two two-vectors is a two-vector, see Figure 2.1.5.

Examples of two-vectors are angular momentum[2]

$$\vec{\vec{L}} = m\,\vec{x} \wedge \vec{v} \tag{2.1.5}$$

and torque

$$\vec{\vec{\tau}} = \frac{d\vec{\vec{L}}}{dt} \tag{2.1.6}$$

[2]Note that $\vec{x}$ is a vector, and hence $\vec{\vec{L}}$ is a two-vector, only in flat space. In contrast, $\vec{dx}$ is a vector, and hence $\vec{v}$ is a vector, even in curved space. Unless otherwise specified, we will assume space is flat.

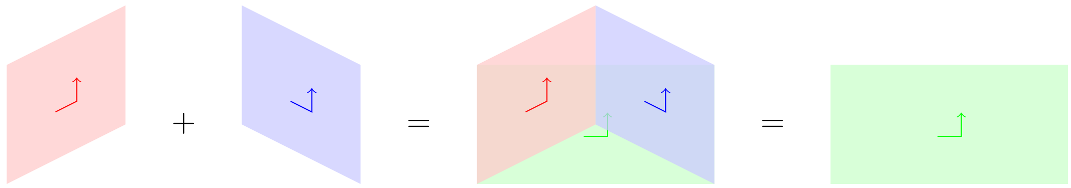

Figure 2.1.5: The sum of two two-vectors is a two-vector.

2.1.2 Differential forms

A **covector** or **one-form** is a tensor that behaves like the local linearized form of contour lines or an extrinsically oriented codimension[3] one plane density, see Figure 2.1.6.

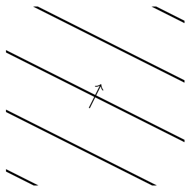

Figure 2.1.6: A one-form.

Comparing vectors and one-forms, the magnitude of a vector is given by its length, while the magnitude of a one-form is given by the density of its planes. The direction of a vector is along its length (intrinsically oriented), while the direction of a one-form is normal to its planes (extrinsically oriented) in the sense that $\underline{n} \cdot \vec{v} = 0$ for any vector $\vec{v}$ lying in the plane of the one-form $\underline{n}$. Thus, a vector is an intrinsically oriented dimension one plane element, while a one-form is an extrinsically oriented codimension one plane density.

A scalar times a one-form is a one-form and the sum of two one-forms is a one-form, see Figure 2.1.7.

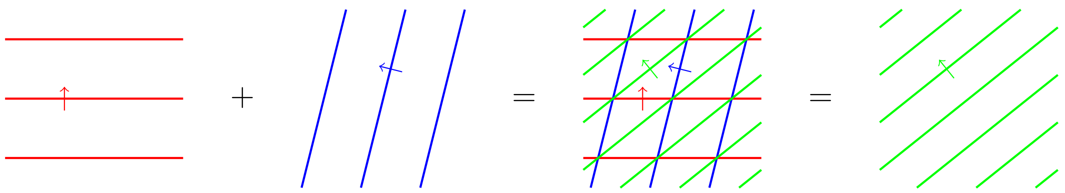

Figure 2.1.7: The sum of two one-forms is a one-form.

Examples of one-forms are wave "vector" $\underline{k}$, electric field

$$\underline{E} = -\underline{\nabla}\phi \equiv -\frac{\partial \phi}{\partial \vec{x}} \tag{2.1.7}$$

[3]Codimension d is dimension $D - d$ where D is the dimension of the space.

and force

$$\underline{F} = q\underline{E} \tag{2.1.8}$$

The exterior product of two one-forms is a **two-form** given by the oriented intersections of the one-form planes, see Figure 2.1.8. Note that the position of the intersections

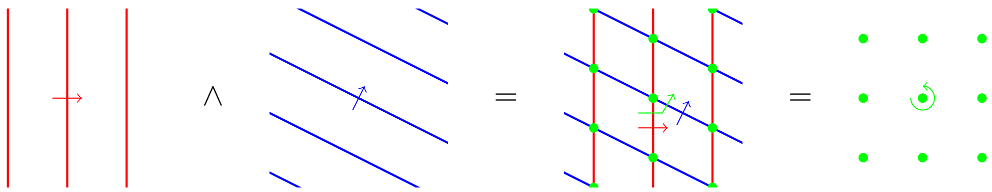

Figure 2.1.8: The exterior product of two one-forms is a two-form.

does not matter, only their density and orientation.

A scalar times a two-form is a two-form and the sum of two two-forms is a two-form too, see Figure 2.1.9.

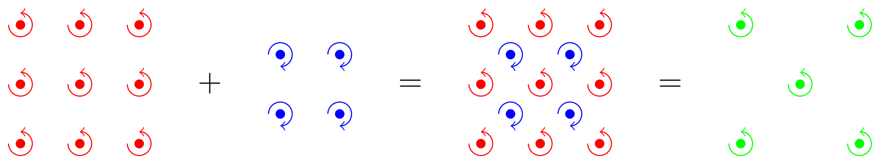

Figure 2.1.9: The sum of two two-forms is a two-form.

Examples of two-forms are magnetic flux

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \tag{2.1.9}$$

and electric current density

$$\underline{\underline{j}} = \underline{\underline{\underline{\rho}}} \cdot \vec{v} \tag{2.1.10}$$

2.1.3 Tensor algebra

A vector can be **contracted** with a one-form to give a scalar, see Figure 2.1.10.

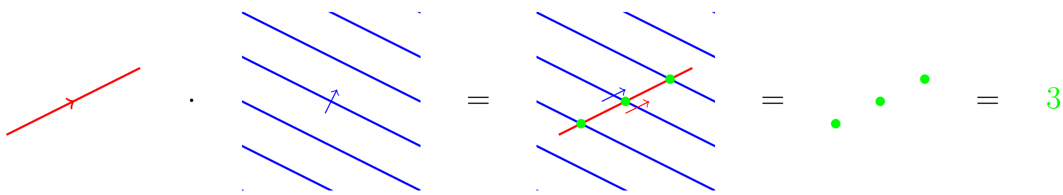

Figure 2.1.10: A vector contracted with a one-form gives a scalar.

Similarly, a two-vector can be contracted with a two-form to give a scalar, see Figure 2.1.11.

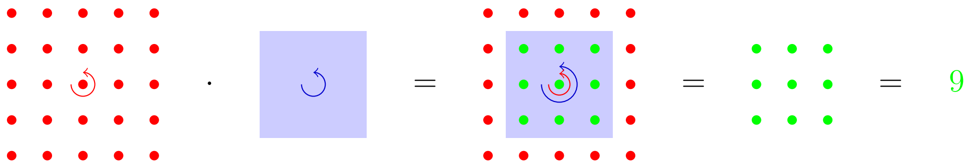

Figure 2.1.11: A two-form contracted with a two-vector gives a scalar.

Other combinations are also straightforward. For example

$$\vec{a} \cdot (\underline{b} \wedge \underline{c}) = (\vec{a} \cdot \underline{b})\, \underline{c} - (\vec{a} \cdot \underline{c})\, \underline{b} \tag{2.1.11}$$

and

$$(\vec{a} \wedge \vec{b}) \cdot (\underline{c} \wedge \underline{d}) = (\vec{a} \cdot \underline{c})\,(\vec{b} \cdot \underline{d}) - (\vec{a} \cdot \underline{d})\,(\vec{b} \cdot \underline{c}) \tag{2.1.12}$$

In general, an n-vector is an intrinsically oriented dimension n plane element, and an n-form is an extrinsically oriented codimension n plane density. The wedge product of an m-vector with an n-vector is an $(m+n)$-vector, and similarly for forms. An m-vector contracted with an n-form is an $(m-n)$-vector or an $(n-m)$-form.

The multivectors and differential forms in three dimensions are shown in Figure 2.1.12

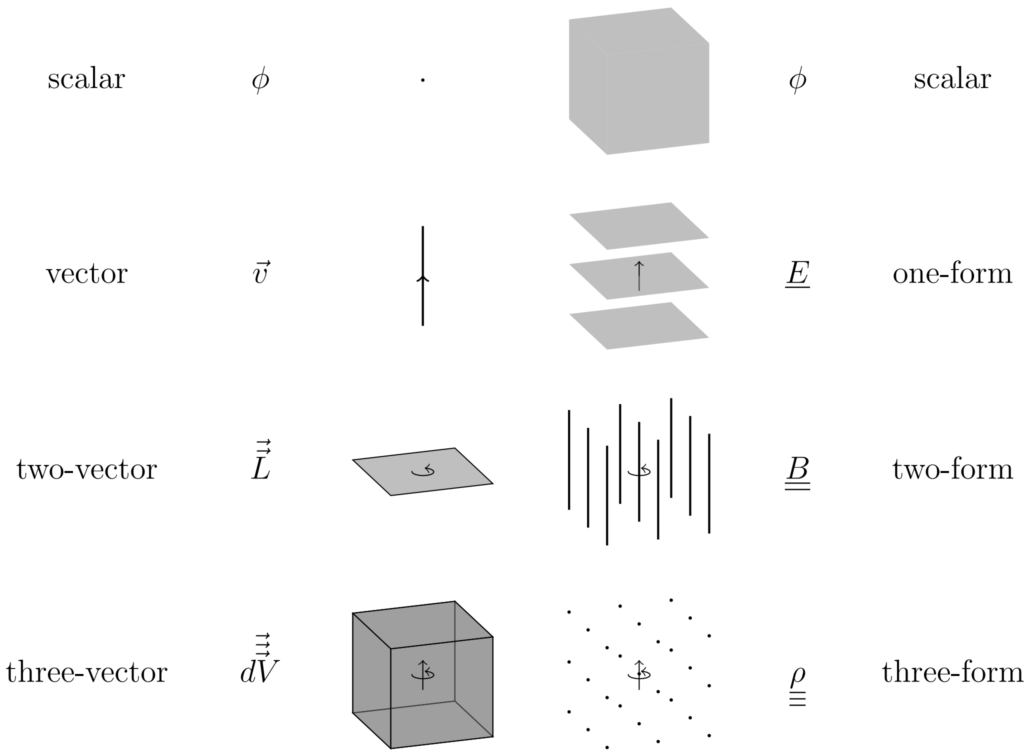

Figure 2.1.12: Multivectors and differential forms in three dimensions.

2.1.4 Volume form and metric

The **volume form** $\boldsymbol{\epsilon}$ and **volume element** $\boldsymbol{\epsilon}^{-1}$ are tensors that measure oriented volumes and densities in a space. They are related by

$$\boldsymbol{\epsilon}^{-1} \cdot \boldsymbol{\epsilon} = 1 \tag{2.1.13}$$

In an N-dimensional space, the volume form is an N-form representing an oriented unit density and the volume element is an N-vector representing an oriented unit volume. They can convert n-vectors into $N-n$ forms and vice versa. For example, in three dimensions, the charge density $\underline{\underline{\underline{\rho}}}$ can be converted into the charge per unit volume ρ

$$\rho = \underline{\underline{\underline{\rho}}} \cdot \vec{\vec{\vec{\epsilon}}} \tag{2.1.14}$$

The **metric** is a tensor that measures lengths and angles in a space. It does not have the elegant topological properties of the multivectors and differential forms of the previous sections but instead introduces geometry.

In this more general context it is often helpful to use the **abstract index notation** in which the nature of a tensor is indicated by the position of indices

$$\begin{aligned} \vec{v} \ &\leftrightarrow\ v^{\mathbf{a}} \quad , \quad \vec{\vec{v}} \ \leftrightarrow\ v^{\mathbf{ab}} = -v^{\mathbf{ba}} \\ \underline{\omega} \ &\leftrightarrow\ \omega_{\mathbf{a}} \quad , \quad \underline{\underline{\omega}} \ \leftrightarrow\ \omega_{\mathbf{ab}} = -\omega_{\mathbf{ba}} \end{aligned} \tag{2.1.15}$$

and contractions are denoted by repeated indices

$$\begin{aligned} u = \vec{v} \cdot \underline{\omega} \ &\leftrightarrow\ u = v^{\mathbf{a}}\omega_{\mathbf{a}} \quad , \quad \vec{u} = \vec{\vec{v}} \cdot \underline{\omega} \ \leftrightarrow\ u^{\mathbf{a}} = v^{\mathbf{ab}}\omega_{\mathbf{b}} \\ \underline{u} = \vec{v} \cdot \underline{\underline{\omega}} \ &\leftrightarrow\ u_{\mathbf{b}} = v^{\mathbf{a}}\omega_{\mathbf{ab}} \quad , \quad u = \vec{\vec{v}} \cdot \underline{\underline{\omega}} \ \leftrightarrow\ u = \tfrac{1}{2!}v^{\mathbf{ab}}\omega_{\mathbf{ab}} \end{aligned} \tag{2.1.16}$$

where the 1/2! arises because two pairs of indices have been contracted. Also

$$\vec{u} \wedge \vec{v} \ \leftrightarrow\ u^{\mathbf{a}} \wedge v^{\mathbf{b}} \ = \ u^{\mathbf{a}}v^{\mathbf{b}} - u^{\mathbf{b}}v^{\mathbf{a}} \tag{2.1.17}$$

The metric

$$g_{\mathbf{ab}} = g_{\mathbf{ba}} \tag{2.1.18}$$

and inverse metric

$$g^{\mathbf{ab}} = g^{\mathbf{ba}} \tag{2.1.19}$$

are related by

$$g^{\mathbf{ab}}g_{\mathbf{bc}} = \delta^{\mathbf{a}}_{\mathbf{c}} \tag{2.1.20}$$

where the identity tensor has the property

$$\delta^{\mathbf{a}}_{\mathbf{b}}v^{\mathbf{b}} = v^{\mathbf{a}} \tag{2.1.21}$$

The metric gives the magnitudes of tensors

$$|\vec{v}|^2 \ = \ g_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}} \quad , \quad |\underline{\omega}|^2 \ = \ g^{\mathbf{ab}}\omega_{\mathbf{a}}\omega_{\mathbf{b}} \tag{2.1.22}$$

and more generally the scalar product

$$\vec{u}\cdot\vec{v} \;=\; g_{\mathbf{ab}}u^{\mathbf{a}}v^{\mathbf{b}} \quad , \quad \underline{\omega}\cdot\underline{\sigma} \;=\; g^{\mathbf{ab}}\omega_{\mathbf{a}}\sigma_{\mathbf{b}} \tag{2.1.23}$$

The scalar product of two vectors can be expressed as

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta \tag{2.1.24}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. The scalar product of two n-vectors or two n-forms can be expressed in exactly the same form.

The metric can also convert an n-vector into an n-form and vice versa

$$v_{\mathbf{a}} \;=\; g_{\mathbf{ab}}v^{\mathbf{b}} \quad , \quad \omega^{\mathbf{a}} \;=\; g^{\mathbf{ab}}\omega_{\mathbf{b}} \tag{2.1.25}$$

The components of the metric can be conveniently seen in terms of the magnitude squared of an infinitesimal displacement

$$ds^2 = \vec{dx}\cdot\vec{dx} = g_{\mathbf{ab}}dx^{\mathbf{a}}dx^{\mathbf{b}} \tag{2.1.26}$$

In Cartesian coordinates (x, y), this takes the Pythagorean form

$$ds^2 = dx^2 + dy^2 \tag{2.1.27}$$

corresponding to the trivial metric components $g_{xx} = g_{yy} = 1$ and $g_{xy} = 0$. Thus, in Cartesian coordinates, n-vectors and n-forms related by the metric as in Eq. (2.1.25) have the same components. In contrast, in polar coordinates (r, θ),

$$ds^2 = dr^2 + r^2\,d\theta^2 \tag{2.1.28}$$

corresponding to the metric components $g_{rr} = 1$, $g_{r\theta} = 0$ and $g_{\theta\theta} = r^2$.

Combining the volume form and metric, in three dimensions, one can define the cross product of two vectors

$$\vec{u}\times\vec{v} \;\leftrightarrow\; g^{\mathbf{ab}}\epsilon_{\mathbf{bcd}}u^{\mathbf{c}}v^{\mathbf{d}} \tag{2.1.29}$$

but why one would want to define something so complex is unclear.