2.2 Framework

2.2.1 Space and time

The central idea of relativity is that space and time are unified into **spacetime**. In general relativity, spacetime is dynamical with spacetime curvature identified with gravity. We will neglect the dynamics of spacetime and assume spacetime is flat, as in special relativity.

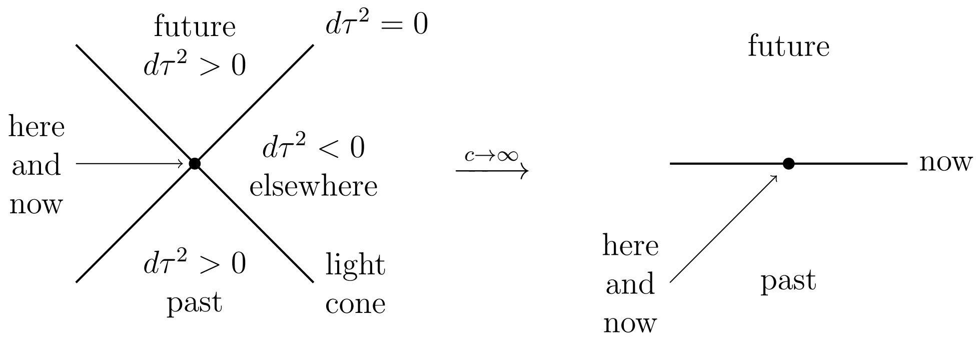

Figure 2.2.1: Relativistic spacetime and its Newtonian limit.

In Minkowski coordinates, an infinitesimal displacement squared can be expressed in terms of the **proper time** τ

$$d\tau^2 = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right) \tag{2.2.1}$$

or equivalently in terms of the **proper distance** s

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \tag{2.2.2}$$

where the minus sign allows us to distinguish time-like and space-like directions. Note that only $d\tau^2$ or ds^2 is physical while dt^2 and $dx^2 + dy^2 + dz^2$ are coordinate dependent. In the Newtonian limit these reduce to

$$d\tau^2\big|_{c\to\infty} = dt^2 \tag{2.2.3}$$

$$ds^2\big|_{dt=0} = \left(dx^2 + dy^2 + dz^2\right)_{dt=0} \tag{2.2.4}$$

See Figure 2.2.1.

2.2.2 Conserved quantities

Our basic principle of mechanics is that a system in a symmetric environment has a corresponding conserved quantity or charge Q. An interaction can transfer charge between subsystems, but the total charge remains constant

$$Q = \sum_i Q_i = \text{constant} \tag{2.2.5}$$

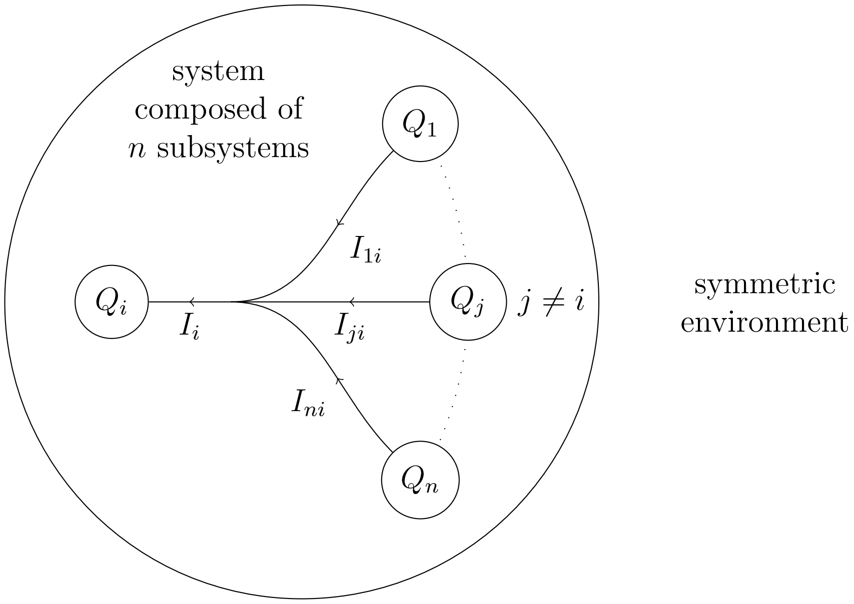

Figure 2.2.2: Continuity equation for a conserved quantity, see Eq. (2.2.7).

where i labels the subsystems. The strength of an interaction can be measured by the rate of transfer, or flow, of charge I_{ij} from subsystem i to subsystem j. By definition, the current in one direction is minus the current in the other

$$I_{ij} = -I_{ji} \tag{2.2.6}$$

The continuity equation then states that the rate of increase of a subsystem's charge is equal to the net flow of charge into the subsystem

$$\frac{dQ_i}{dt} = I_i = \sum_{j \neq i} I_{ji} \tag{2.2.7}$$

See Figure 2.2.2.

Three important cases of this principle are:

Space A system in a spatially homogeneous environment has a conserved quantity called **momentum** (Newton's first law)

$$p = \sum_i p_i = \text{constant} \tag{2.2.8}$$

The **force** of an interaction between subsystems is the rate of transfer, or flow, of momentum. The force from subsystem i to subsystem j is by definition minus the force from subsystem j to subsystem i (Newton's third law)

$$F_{ij} = -F_{ji} \tag{2.2.9}$$

The continuity equation for momentum then states that the rate of increase of a subsystem's momentum is equal to the net flow of momentum into the subsystem,

Symmetry	Charge	Current	Continuity equation
temporal translation	energy	power	$\frac{dE}{dt} = P$
spatial translation	momentum	force	$\frac{dp}{dt} = F$
spatial rotation	angular momentum	torque	$\frac{dL}{dt} = \tau$

Table 2.2.1: The laws governing energy, momentum and angular momentum are simple consequences of their conservation.

i.e. the net force, (Newton's second law)

$$\frac{dp_i}{dt} = F_i = \sum_{j \neq i} F_{ji} \tag{2.2.10}$$

Time A system in a temporally homogeneous environment has a conserved quantity called **energy**

$$E = \sum_i E_i = \text{constant} \tag{2.2.11}$$

The **power** of an interaction is the rate of transfer, or flow, of energy

$$P_{ij} = -P_{ji} \tag{2.2.12}$$

The continuity equation for energy is

$$\frac{dE_i}{dt} = P_i = \sum_{j \neq i} P_{ji} \tag{2.2.13}$$

Angle A system in an isotropic environment has a conserved quantity called **angular momentum**

$$L = \sum_i L_i = \text{constant} \tag{2.2.14}$$

The **torque** of an interaction is the rate of transfer, or flow, of angular momentum

$$\tau_{ij} \equiv -\tau_{ji} \tag{2.2.15}$$

The continuity equation for angular momentum is

$$\frac{dL_i}{dt} = \tau_i = \sum_{j \neq i} \tau_{ji} \tag{2.2.16}$$

2.2.3 Lagrangian mechanics

The **Lagrangian**

$$L = L\left(\dot{\vec{q}}, q, t\right) \tag{2.2.17}$$

determines the physics of a system via **Lagrange's equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) = \frac{\partial L}{\partial \vec{q}} \tag{2.2.18}$$

This equation allows us to identify the **momentum**

$$\underline{p} = \frac{\partial L}{\partial \dot{\vec{q}}} \tag{2.2.19}$$

and the **force** applied to the system

$$\underline{F} = \frac{\partial L}{\partial \vec{q}} \tag{2.2.20}$$

Using Eq. (2.2.18) and the chain rule

$$\begin{aligned}
\frac{d}{dt}\left(\dot{\vec{q}} \cdot \frac{\partial L}{\partial \dot{\vec{q}}} - L\right) &= \frac{d\dot{\vec{q}}}{dt} \cdot \frac{\partial L}{\partial \dot{\vec{q}}} + \dot{\vec{q}} \cdot \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) - \frac{dL}{dt} && (2.2.21) \\
&= \frac{d\dot{\vec{q}}}{dt} \cdot \frac{\partial L}{\partial \dot{\vec{q}}} + \frac{d\vec{q}}{dt} \cdot \frac{\partial L}{\partial \vec{q}} - \frac{dL}{dt} && (2.2.22) \\
&= -\frac{\partial L}{\partial t} && (2.2.23)
\end{aligned}$$

which allows us to identify the **energy**

$$E = \dot{\vec{q}} \cdot \frac{\partial L}{\partial \dot{\vec{q}}} - L \tag{2.2.24}$$

and the **power** applied to the system

$$P = -\frac{\partial L}{\partial t} \tag{2.2.25}$$

A more fundamental quantity is the **action**

$$S[q(t)] = \int L\left(\dot{\vec{q}}, q, t\right) dt \tag{2.2.26}$$

Lagrange's equation, Eq. (2.2.18), can be derived from **Hamilton's principle**

$$\frac{\delta S}{\delta q(t)} = 0 \tag{2.2.27}$$

which in turn can be derived as the $\hbar \to 0$ limit of the path integral formulation of quantum mechanics.

2.2.4 Hamiltonian mechanics

The **Hamiltonian** is related to the Lagrangian by

$$H\big(\underline{p}, q, t\big) = \underline{p} \cdot \vec{\dot{q}} - L\Big(\vec{\dot{q}}, q, t\Big) \tag{2.2.28}$$

Hamilton's equations

$$\frac{\vec{dq}}{dt} = \frac{\partial H}{\partial \underline{p}} \tag{2.2.29}$$

$$\frac{d\underline{p}}{dt} = -\frac{\partial H}{\vec{\partial q}} \tag{2.2.30}$$

are equivalent to Lagrange's equation and again allow us to identify the force applied to the system

$$\underline{F} = -\frac{\partial H}{\vec{\partial q}} \tag{2.2.31}$$

Using the chain rule and Eqs. (2.2.29) and (2.2.30)

$$\frac{dH}{dt} = \frac{\partial H}{\partial \underline{p}} \cdot \frac{d\underline{p}}{dt} + \frac{\partial H}{\vec{\partial q}} \cdot \frac{\vec{dq}}{dt} + \frac{\partial H}{\partial t} \tag{2.2.32}$$

$$= -\frac{\partial H}{\partial \underline{p}} \cdot \frac{\partial H}{\vec{\partial q}} + \frac{\partial H}{\vec{\partial q}} \cdot \frac{\partial H}{\partial \underline{p}} + \frac{\partial H}{\partial t} \tag{2.2.33}$$

$$= \frac{\partial H}{\partial t} \tag{2.2.34}$$

which allows us to identify the energy

$$E = H \tag{2.2.35}$$

and the power applied to the system

$$P = \frac{\partial H}{\partial t} \tag{2.2.36}$$