2.3 Materials

2.3.1 Newtonian particle and field

A **Newtonian particle** is something that exists at a single point in space, and so its motion can be described by its position as a function of time, $q(t)$. A particle has energy, momentum and angular momentum. The energy of a particle is called its **kinetic energy**, K, because it is due to the motion of the particle.

A **Newtonian field** is something that has a value at every point in space, and so its motion can be described by its value as a function of space and time, $\phi(q,t)$. In relativity, fields are dynamical and have energy, momentum and angular momentum. However, in the Newtonian limit, $c \to \infty$, fields become non-dynamical, with their state being determined by the particles they interact with. In this limit, they cannot store momentum or angular momentum, instead transferring it instantaneously between particles, but can store energy. [1] The energy stored in the fields becomes a function of the particle positions, $V(q_1, \ldots, q_n)$, and is called the **potential energy**.

	Newtonian	
Charge	particle	field
energy	K	V
momentum	p	neglected
angular momentum	L	neglected

Table 2.3.1: Energy, momentum and angular momentum of a Newtonian particle and field.

The Lagrangian for a Newtonian particle and field is

$$L = \frac{1}{2} m g_{ab} \dot{q}^a \dot{q}^b - V(q,t) \tag{2.3.1}$$

where g_{ab} is the spatial metric. Using Eq. (2.2.19), the momentum of the system, and hence the particle since the Newtonian field stores no momentum, is

$$p_a = m g_{ab} \dot{q}^b \tag{2.3.2}$$

For example, in tensor form

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \dot{q}^{\mathbf{b}} \tag{2.3.3}$$

while in Cartesian coordinates $g_{xx} = 1$ so

$$p_x = m \dot{x} \tag{2.3.4}$$

[1] Note that a spring whose mass is neglected acts in the same way as a field.

and in polar coordinates $g_{rr} = 1$, $g_{r\theta} = g_{\theta r} = 0$ and $g_{\theta\theta} = r^2$ so

$$p_r = m\dot{r} \tag{2.3.5}$$
$$p_\theta = mr^2\dot{\theta} \tag{2.3.6}$$

Using Eq. (2.2.20), the force exerted on the system, and hence the particle, is

$$F_a = -\frac{\partial V}{\partial q^a} + \frac{1}{2}m\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c \tag{2.3.7}$$

For example, in tensor form the metric is regarded as constant giving

$$F_{\mathbf{a}} = -\frac{\partial V}{\partial q^{\mathbf{a}}} \tag{2.3.8}$$

while in Cartesian coordinates

$$F_x = -\frac{\partial V}{\partial x} \tag{2.3.9}$$

and in polar coordinates

$$F_r = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 \tag{2.3.10}$$
$$F_\theta = -\frac{\partial V}{\partial \theta} \tag{2.3.11}$$

where $mr\dot{\theta}^2$ is the centrifugal force.

Using Eq. (2.2.24), the energy of the system is

$$E = \frac{1}{2}mg_{ab}\dot{q}^a\dot{q}^b + V(q,t) \tag{2.3.12}$$

where the first term is the energy of the particle and the second is the energy of the field. Using Eq. (2.2.25), the chain rule and Eq. (2.3.7), the power applied to the system is

$$P = \frac{\partial V}{\partial t} \tag{2.3.13}$$
$$= \frac{dV}{dt} - \frac{\partial V}{\partial q^a}\dot{q}^a \tag{2.3.14}$$
$$= \frac{dV}{dt} + F_a\dot{q}^a - \frac{1}{2}m\frac{\partial g_{bc}}{\partial q^a}\dot{q}^a\dot{q}^b\dot{q}^c \tag{2.3.15}$$

where the first term is the power applied to the field and the other terms are the power applied to the particle. For example, in tensor form

$$P = \frac{dV}{dt} + F_{\mathbf{a}}\dot{q}^{\mathbf{a}} \tag{2.3.16}$$

while in Cartesian coordinates

$$P = \frac{dV}{dt} + F_x\dot{x} \tag{2.3.17}$$

and in polar coordinates

$$P = \frac{dV}{dt} + \left(F_r - mr\dot{\theta}^2\right)\dot{r} + F_\theta\dot{\theta} \tag{2.3.18}$$

Lagrange's equation, Eq. (2.2.18), gives

$$\frac{d}{dt}\left(mg_{ab}\dot{q}^b\right) = F_a \tag{2.3.19}$$

For example, in tensor form

$$mg_{\mathbf{ab}}\ddot{q}^{\mathbf{b}} = F_{\mathbf{a}} \tag{2.3.20}$$

while in Cartesian coordinates

$$m\ddot{x} = F_x \tag{2.3.21}$$

and in polar coordinates

$$m\ddot{r} = F_r \tag{2.3.22}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = F_\theta \tag{2.3.23}$$

The Hamiltonian, Eq. (2.2.28), is

$$H = \frac{1}{2m}g^{ab}p_a p_b + V(q,t) \tag{2.3.24}$$

Using Eq. (2.2.31), the force exerted on the system, and hence the particle, is

$$F_a = -\frac{\partial V}{\partial q^a} - \frac{1}{2m}\frac{\partial g^{bc}}{\partial q^a}p_b p_c \tag{2.3.25}$$

Hamilton's equations, Eqs. (2.2.29) and (2.2.30), give

$$\frac{dq^a}{dt} = \frac{1}{m}g^{ab}p_b \tag{2.3.26}$$

$$\frac{dp_a}{dt} = F_a \tag{2.3.27}$$

2.3.2 Newtonian particles

Consider a system of Newtonian particles with masses m_i, positions $x_i(t)$ and forces acting on the particles $F_i(t)$. If we decompose the particle positions into

$$x_i = x_{\text{CM}} + \delta x_i \tag{2.3.28}$$

where the **center of mass**

$$x_{\text{CM}} = \frac{1}{m}\sum_i m_i x_i \tag{2.3.29}$$

and the δx_i are the internal displacements, then we can decompose other physical quantities similiarly:

mass

$$m = \sum_i m_i = m_{\mathrm{CM}} \tag{2.3.30}$$

momentum

$$p = \sum_i p_i = \sum_i m_i \dot{x}_i = m\dot{x}_{\mathrm{CM}} = p_{\mathrm{CM}} \tag{2.3.31}$$

force

$$F = \sum_i F_i = \sum_i m_i \ddot{x}_i = m\ddot{x}_{\mathrm{CM}} = F_{\mathrm{CM}} \tag{2.3.32}$$

kinetic energy

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m\dot{x}_{\mathrm{CM}}^2 + \sum_i \frac{1}{2} m_i \, \delta\dot{x}_i^2 = K_{\mathrm{CM}} + K_{\mathrm{int}} \tag{2.3.33}$$

power

$$P = \sum_i P_i = \sum_i F_i \dot{x}_i = F\dot{x}_{\mathrm{CM}} + \sum_i F_i \, \delta\dot{x}_i = P_{\mathrm{CM}} + P_{\mathrm{int}} \tag{2.3.34}$$

angular momentum

$$L = \sum_i L_i = \sum_i m_i x_i \wedge \dot{x}_i = m x_{\mathrm{CM}} \wedge \dot{x}_{\mathrm{CM}} + \sum_i m_i \, \delta x_i \wedge \delta\dot{x}_i = L_{\mathrm{CM}} + L_{\mathrm{int}} \tag{2.3.35}$$

torque

$$\tau = \sum_i \tau_i = \sum_i x_i \wedge F_i = x_{\mathrm{CM}} \wedge F + \sum_i \delta x_i \wedge F_i = \tau_{\mathrm{CM}} + \tau_{\mathrm{int}} \tag{2.3.36}$$

2.3.3 Newtonian rigid body

For a Newtonian **rigid body** rotating about the origin, the radial distance of each particle in the body is constant and the angular velocity about the origin is the same for all particles

$$\dot{r}_i = 0 \tag{2.3.37}$$
$$\dot{\theta}_i = \dot{\theta} \tag{2.3.38}$$

Therefore

$$L = \sum_i m_i r_i^2 \dot{\theta}_i = I\dot{\theta} \tag{2.3.39}$$

and

$$K = \sum_i \frac{1}{2} m_i r_i^2 \dot{\theta}_i^2 = \frac{1}{2} I \dot{\theta}^2 \tag{2.3.40}$$

where the **moment of inertia** of the body

$$I = \sum_i m_i r_i^2 \tag{2.3.41}$$

is constant. Taking derivatives we get

$$\tau = I\ddot{\theta} \tag{2.3.42}$$

and

$$P = \tau\dot{\theta} \tag{2.3.43}$$

For example, the moment of inertia of a uniform rod of mass M and length $2R$ rotating about its center is

$$I = \int_0^R \frac{M}{2R} r^2 \cdot 2\,dr = \frac{1}{3}MR^2 \tag{2.3.44}$$

of a unifirm disc of mass M and radius R rotating in the plane of the disc and about its center is

$$I = \int_0^R \frac{M}{\pi R^2} r^2 \cdot 2\pi r \cdot dr = \frac{1}{2}MR^2 \tag{2.3.45}$$

and of a uniform ball of mass M and radius R rotating about its center is

$$I = \int_0^R \frac{M}{\frac{4}{3}\pi R^3} r^2 \cdot 2\pi r \cdot 2\sqrt{R^2 - r^2} \cdot dr = \frac{2}{5}MR^2 \tag{2.3.46}$$

2.3.4 Relativistic particle

A **relativistic particle** is something that exists as a **worldline** in spacetime. The action for a relativistic particle is proportional to its length

$$\begin{aligned} S &= -mc^2 \int d\tau && (2.3.47) \\ &= -mc^2 \int \sqrt{g_{ab} dx^a dx^b} && (2.3.48) \\ &= -mc^2 \int \sqrt{dt^2 - \frac{1}{c^2} h_{ab} dq^a dq^b} && (2.3.49) \\ &= -mc^2 \int \sqrt{1 - \frac{1}{c^2} h_{ab} \frac{dq^a}{dt} \frac{dq^b}{dt}}\, dt && (2.3.50) \end{aligned}$$

where g_{ab} is the spacetime metric and h_{ab} is the spatial metric. Eq. (2.2.19) gives

$$\begin{aligned} p_a &= -mc^2 \frac{\partial}{\partial \dot{q}^a} \sqrt{1 - \frac{1}{c^2} h_{bc} \dot{q}^b \dot{q}^c} && (2.3.51) \\ &= \frac{m h_{ab} \dot{q}^b}{\sqrt{1 - \frac{1}{c^2} h_{cd} \dot{q}^c \dot{q}^d}} && (2.3.52) \\ &= m h_{ab} \frac{dq^b}{d\tau} && (2.3.53) \end{aligned}$$

and Eq. (2.2.24) gives [2]

$$E = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} \tag{2.3.54}$$

$$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2.3.55}$$

$$= mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{mv^4}{c^2}\right) \tag{2.3.56}$$

where $v^2 = h_{ab}\dot{q}^a\dot{q}^b$. Combining Eqs. (2.3.52) and (2.3.55) gives

$$E^2 - p^2c^2 = m^2c^4 \tag{2.3.57}$$

Thus the energy of a massless particle is

$$E = pc \tag{2.3.58}$$

[2] The m in the famous equation $E = mc^2$ is defined by the Newtonian formula for the momentum, Eq. (2.3.2), and not as the constant in Eq. (2.3.47), and so depends on the speed.