2.4 Interactions

2.4.1 Empirical interactions

Normal force

This prevents one object moving into another. It acts at right angles to the surfaces in contact and has a magnitude F_{N} which balances any forces pushing the surfaces together.

Static friction

This resists sliding of two surfaces in static contact. It acts parallel to the surfaces in contact and has a magnitude which balances any forces trying to make the surfaces slide past each other, up to a maximum strength proportional to the normal force between the surfaces

$$F_{\mathrm{s}} \leq \mu_{\mathrm{s}} F_{\mathrm{N}} \tag{2.4.1}$$

Kinetic friction

This opposes the sliding motion of two surfaces in sliding contact. It acts parallel to the surfaces in contact and has a magnitude proportional to the normal force

$$F_{\mathrm{k}} = \mu_{\mathrm{k}} F_{\mathrm{N}} \tag{2.4.2}$$

and less than or equal to the maximum static friction

$$\mu_{\mathrm{k}} \leq \mu_{\mathrm{s}} \tag{2.4.3}$$

2.4.2 Effective interactions

A potential $V(x)$ can be expanded in a Taylor series about any non-singular point x_0

$$V(x_0 + \delta x) = V_0 + V_0' \,\delta x + \frac{1}{2} V_0'' \,\delta x^2 + \mathcal{O}\left[(\delta x)^3\right] \tag{2.4.4}$$

with the higher order terms being small if δx is sufficiently small. The constant first term has no effect on the dynamics and so can be ignored. Thus the second term will typically dominate giving

$$V = V_0 + V_0' \,\delta x + \mathcal{O}\left[(\delta x)^2\right] \tag{2.4.5}$$

and

$$F = -V_0' + \mathcal{O}(\delta x) \tag{2.4.6}$$

For example, the **local gravitational potential** due to a particle of mass m near the surface of the Earth is

$$V = V_{\oplus} + mg \,\delta r + \mathcal{O}\left[(\delta r)^2\right] \tag{2.4.7}$$

and the corresponding force on the particle is

$$F = -mg + \mathcal{O}(\delta r) \tag{2.4.8}$$

where $V_\oplus$ is the potential energy at the Earth's surface and $\delta r = r - r_\oplus$ is the height above the Earth's surface.

If $V_0' = 0$ then there is no force at x_0, i.e. x_0 is a point of **equilibrium**. Using Eq. (2.4.4), near a point of equilibrium

$$V = V_0 + \frac{1}{2} V_0''\,\delta x^2 + \mathcal{O}\left[(\delta x)^3\right] \tag{2.4.9}$$

and

$$F = -V_0''\,\delta x + \mathcal{O}\left[(\delta x)^2\right] \tag{2.4.10}$$

$V_0'' > 0$ corresponds to stable equilibrium and $V_0'' < 0$ corresponds to unstable equilibrium. For example, oscillation, vibration, Hooke's law, etc. The approximate equation of motion

$$\ddot{\delta x} = -\frac{V_0''}{m}\,\delta x \tag{2.4.11}$$

gives rise to **simple harmonic motion**

$$\delta x = A \sin\left(\sqrt{\frac{V_0''}{m}}\,t\right) + B \cos\left(\sqrt{\frac{V_0''}{m}}\,t\right) \tag{2.4.12}$$

2.4.3 Fundamental interactions

The Lagrangian for **electrostatics** is

$$L(\underline{\nabla}\phi, \phi, x) = \frac{1}{2}\varepsilon_0 \int \underline{\nabla}\phi \cdot \underline{\nabla}\phi \sqrt{g(x)}\,d^3x - \int \phi\rho \sqrt{g(x)}\,d^3x \tag{2.4.13}$$

where ϕ is the electric potential, $\underline{E} = -\underline{\nabla}\phi$ is the electric field, ρ is the charge density, ε_0 is the vacuum permittivity, and g is the determinant of the metric components so that $\sqrt{g(x)}\,d^3x$ is the infinitesimal physical volume. For example, in Cartesian coordinates $g = g_{xx}g_{yy}g_{zz} = 1$ so

$$\sqrt{g(x)}\,d^3x = dx\,dy\,dz \tag{2.4.14}$$

and in spherical polar coordinates $g = g_{rr}g_{\theta\theta}g_{\varphi\varphi} = 1 \cdot r^2 \cdot r^2 \sin^2\theta$ so

$$\sqrt{g(x)}\,d^3x = r^2 \sin\theta\,dr\,d\theta\,d\varphi \tag{2.4.15}$$

Lagrange's equation is

$$-\varepsilon_0 \underline{\nabla} \cdot \underline{\nabla}\phi = \rho \tag{2.4.16}$$

which has solution

$$\phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(y)}{|\vec{x} - \vec{y}|} \sqrt{g(y)}\,d^3y \tag{2.4.17}$$

Thus the field is determined by the charge density and so is not dynamical, as expected in Newtonian physics.

The field energy is

$$V = \frac{1}{2}\varepsilon_0 \int \underline{\nabla}\phi \cdot \underline{\nabla}\phi \, \sqrt{g(x)}\, d^3x \tag{2.4.18}$$

$$= \frac{1}{2} \int \phi(x)\, \rho(x)\, \sqrt{g(x)}\, d^3x \tag{2.4.19}$$

$$= \frac{1}{8\pi\varepsilon_0} \int \frac{\rho(x)\,\rho(y)}{|\vec{x}-\vec{y}|} \sqrt{g(x)}\, d^3x \, \sqrt{g(y)}\, d^3y \tag{2.4.20}$$

where we have used integration by parts and Eqs. (2.4.16) and (2.4.17). In the case of two particles of charge q_1 and q_2 at positions x_1 and x_2, Eq. (2.4.20) reduces to

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} \tag{2.4.21}$$

Therefore the force exerted on particle one is

$$\underline{F}_1 = -\frac{\partial V}{\partial \vec{x}_1} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \frac{\underline{x}_1 - \underline{x}_2}{|\vec{x}_1 - \vec{x}_2|} \tag{2.4.22}$$

and the force exerted on particle two is

$$\underline{F}_2 = -\frac{\partial V}{\partial \vec{x}_2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \frac{\underline{x}_2 - \underline{x}_1}{|\vec{x}_1 - \vec{x}_2|} \tag{2.4.23}$$

Thus $\underline{F}_1 + \underline{F}_2 = 0$ and so the field stores no momentum, as expected in Newtonian physics.

Newtonian gravity has the same form with ϕ the gravitational potential, ρ the mass density, q the mass m and $1/\varepsilon_0 \to -4\pi G$ where G is the gravitational constant.