Final Exam - 10am Saturday 14th June, Creative 201

You should start from basic principles and give reasoning at every step. Answers should be clear and concise. Unclear or excessively long answers will not be graded.

Q1. Using

$$\langle\theta,\phi|\,\hat{L}_{12}\,|\psi\rangle \;=\; -i\hbar\,\frac{\partial}{\partial\phi}\,\psi(\theta,\phi) \qquad \text{(Q1.1)}$$

$$\langle\theta,\phi|\,\hat{L}_{\pm}\,|\psi\rangle \;=\; \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)\psi(\theta,\phi) \qquad \text{(Q1.2)}$$

and the properties of $\hat{L}_{12}$ and $\hat{L}_{\pm}$ derived for homework, determine the angular momentum eigenfunctions $\psi_{l,m}(\theta,\phi)$.

A1. Using Eqs. (A4.2.2), (A4.2.15) and (Q1.1)

$$0=\langle\theta,\phi|\left(\hat{L}_{12}-m\hbar\right)|l,m\rangle=\hbar\left(-i\,\frac{\partial}{\partial\phi}-m\right)\psi_{l,m}(\theta,\phi) \qquad \text{(A1.1)}$$

therefore

$$\psi_{l,m}(\theta,\phi)\propto e^{im\phi} \qquad \text{(A1.2)}$$

Using Eqs. (A4.2.9), (A4.2.14), (A4.2.15), (Q1.2) and (A1.2)

$$0 \;=\; \langle\theta,\phi|\,\hat{L}_{\pm}\,|l,\pm l\rangle \;=\; \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)\psi_{l,\pm l}(\theta,\phi) \qquad \text{(A1.3)}$$

$$= \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}-l\cot\theta\right)\psi_{l,\pm l}(\theta,\phi) \qquad \text{(A1.4)}$$

therefore

$$\psi_{l,\pm l}(\theta,\phi)\propto\sin^{l}\theta \qquad \text{(A1.5)}$$

Using Eqs. (A4.2.5), (A1.2) and (A1.5)

$$\psi_{l,m}(\theta,\phi) \;\propto\; \langle\theta,\phi|\left(\hat{L}_{\pm}\right)^{l\pm m}|l,\mp l\rangle \qquad \text{(A1.6)}$$

$$= \left[\pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)\right]^{l\pm m}\psi_{l,\mp l}(\theta,\phi) \qquad \text{(A1.7)}$$

$$\propto \left[\pm e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)\right]^{l\pm m}e^{\mp il\phi}\sin^{l}\theta \qquad \text{(A1.8)}$$

$$= \left(\mp\sin^{\mp i\frac{\partial}{\partial\phi}}\theta\, e^{\pm i\phi}\,\frac{\partial}{\partial\cos\theta}\,\sin^{\pm i\frac{\partial}{\partial\phi}}\theta\right)^{l\pm m}e^{\mp il\phi}\sin^{l}\theta \qquad \text{(A1.9)}$$

$$= (\mp1)^{l\pm m}e^{im\phi}\sin^{\pm m}\theta\left(\frac{\partial}{\partial\cos\theta}\right)^{l\pm m}\sin^{2l}\theta \qquad \text{(A1.10)}$$

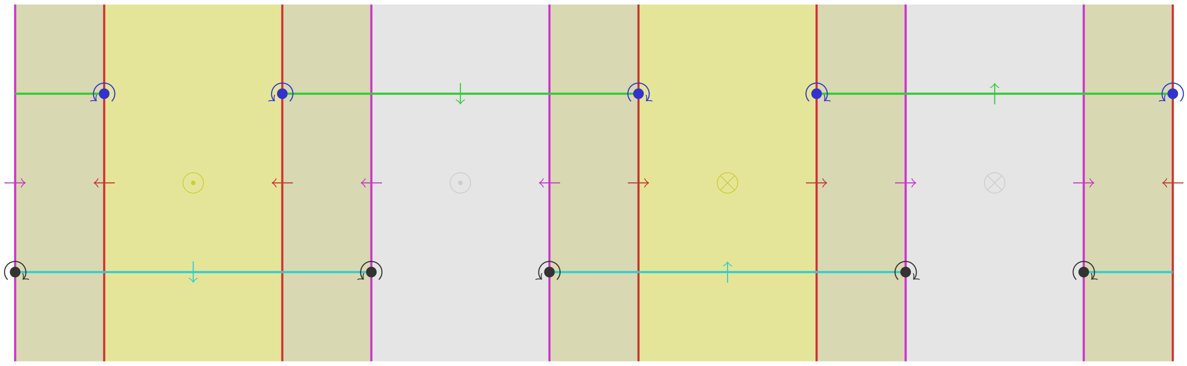

Figure A2.1: An electromagnetic wave. $\underline{\nabla} \wedge \underline{E} = -\dot{\underline{\underline{B}}} = -*\dot{\underline{H}}$, $*\underline{E} = \underline{\underline{D}}$, $*\dot{\underline{E}} = \dot{\underline{\underline{D}}} = \underline{\nabla} \wedge \underline{H}$, $*\underline{H} = \underline{\underline{B}}$.

Q2. Complete the diagram of electromagnetic fields and fluxes in vacuum and identify the energy density and energy flux density. Interpret your diagram.

A2. See Figure A2.1.

$$\underline{\underline{\underline{u}}} = \frac{1}{2}\underline{E} \wedge \underline{\underline{D}} + \frac{1}{2}\underline{H} \wedge \underline{\underline{B}} \quad \text{(A2.1)}$$

$$\underline{\underline{S}} = \underline{E} \wedge \underline{H} \quad \text{(A2.2)}$$