Homework 2 - Quantities and values

Q2.1. Let

$$\langle x|x\rangle = \langle y|y\rangle = 1 \tag{Q2.1.1}$$

$$\langle x|y\rangle = 0 \tag{Q2.1.2}$$

and

$$L_{xy} = i\,|y\rangle\,\langle x| - i\,|x\rangle\,\langle y| \tag{Q2.1.3}$$

Determine the eigenvalues and eigenspaces of L_{xy} and show that they are orthogonal and complete.

A2.1.

$$L_{xy}\left(\alpha\,|x\rangle + \beta\,|y\rangle\right) = \left(i\,|y\rangle\,\langle x| - i\,|x\rangle\,\langle y|\right)\left(\alpha\,|x\rangle + \beta\,|y\rangle\right) = -i\beta\,|x\rangle + i\alpha\,|y\rangle \tag{A2.1.1}$$

therefore for an eigenvector with eigenvalue λ we require

$$\frac{-i\beta}{\alpha} = \frac{i\alpha}{\beta} = \lambda \tag{A2.1.2}$$

which gives $\beta = \pm i\alpha$ and $\lambda = \pm 1$, respectively. Thus the eigenvalues of L_{xy} are 1 and -1 with eigenspaces containing vectors proportional to

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i\,|y\rangle\right) \tag{A2.1.3}$$

$$|-1\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i\,|y\rangle\right) \tag{A2.1.4}$$

respectively. Noting that

$$\langle -1|1\rangle = \frac{1}{2}\left(|x\rangle - i\,|y\rangle\right)^\dagger\left(|x\rangle + i\,|y\rangle\right) = \frac{1}{2}\left(\langle x| + i\,\langle y|\right)\left(|x\rangle + i\,|y\rangle\right) = 0 \tag{A2.1.5}$$

and that $|1\rangle$ and $|-1\rangle$ are two independent kets in a two dimensional Hilbert space, we see that the eigenspaces are orthogonal and complete.

Q2.2. Express $|x\rangle$ and $|y\rangle$ in terms of the eigenvectors of L_{xy}. Calculate $\langle x|\,L_{xy}\,|x\rangle$ and $\langle x|\,L_{xy}^2\,|x\rangle$.

A2.2.

$$|x\rangle = \frac{1}{2}\left(|x\rangle + i\,|y\rangle\right) + \frac{1}{2}\left(|x\rangle - i\,|y\rangle\right) = \frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right) \tag{A2.2.1}$$

$$|y\rangle = \frac{1}{2i}\left(|x\rangle + i\,|y\rangle\right) - \frac{1}{2i}\left(|x\rangle - i\,|y\rangle\right) = \frac{1}{\sqrt{2}\,i}\left(|1\rangle - |-1\rangle\right) \tag{A2.2.2}$$

$$\langle x|\,L_{xy}\,|x\rangle = \frac{1}{2}\left(\langle 1| + \langle -1|\right)L_{xy}\left(|1\rangle + |-1\rangle\right) = \frac{1}{2}\left[1 + (-1)\right] = 0 \tag{A2.2.3}$$

$$\langle x|\,L_{xy}^2\,|x\rangle = \frac{1}{2}\left(\langle 1| + \langle -1|\right)L_{xy}^2\left(|1\rangle + |-1\rangle\right) = \frac{1}{2}\left[1^2 + (-1)^2\right] = 1 \tag{A2.2.4}$$