Chapter 2

Electrodynamics

2.1 Tensor fields

A **tensor field** is something that takes tensor values at every point in a space, usually relativistic spacetime or Newtonian space and time. Tensor fields of the same type can be added, and multiplied by a scalar, in the usual way. There is also a rich calculus relating different types of tensor fields.

2.1.1 Exterior derivative

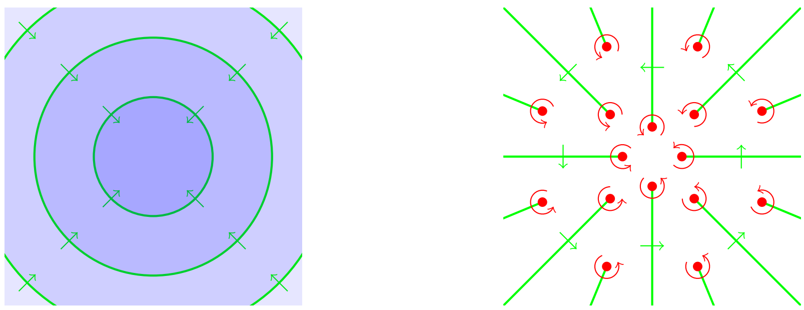

Figure 2.1.1: Left: a scalar (zero-form) field gives rise to a covector (one-form) field; right: a covector (one-form) field gives rise to a two-form field.

The relations in Figure 2.1.1 can be mathematically represented as

$$\underline{\nabla} \wedge \phi = \underline{\xi} \tag{2.1.1}$$

and

$$\underline{\nabla} \wedge \underline{\zeta} = \underline{\underline{\rho}} \tag{2.1.2}$$

where the **exterior derivative** $\underline{\nabla}\wedge$ has the meaning 'the oriented boundaries of' and gives a measure of the spacial rate of change of the tensor field.

The exterior derivative has the important property

$$\boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \boldsymbol{\omega} = 0 \tag{2.1.3}$$

for any differential form $\boldsymbol{\omega}$, since the boundary of a boundary is zero, as can be seen from Figure 2.1.1.

2.1.2 Integration

An n-form field $\boldsymbol{\omega}$ naturally contracts with an n-dimensional surface S to give a scalar

$$\int_S \boldsymbol{\omega} = \text{scalar} \tag{2.1.4}$$

with the same interpretation as the contraction of an n-form with an n-vector. If we divide the surface S into infinitesimal surface elements $\boldsymbol{dS}$, the integral of $\boldsymbol{\omega}$ over S can be written in the more familiar form

$$\int_S \boldsymbol{\omega} \cdot \boldsymbol{dS} \tag{2.1.5}$$

2.1.3 Stokes' theorem

Stokes' theorem states that

$$\int_S \boldsymbol{\nabla} \wedge \boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} \tag{2.1.6}$$

where ∂S is the boundary of S.

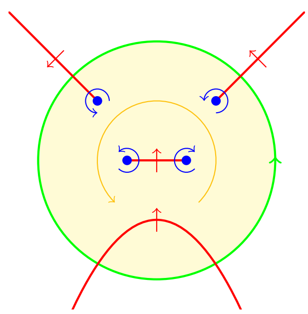

Figure 2.1.2: Stokes' theorem: $\int_S \boldsymbol{\nabla} \wedge \boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} = 2$.