Final Exam - 1pm Sunday 10th December, Creative 412

Your answers should be clear and concise. They should start from basic principles and proceed logically. You may cite material from the lecture notes and homework answers.

Q1. Use diagrammatic methods to show that

$$\vec{\vec{v}} \cdot (\underline{\omega} \wedge \underline{\sigma}) = (\vec{\vec{v}} \cdot \underline{\sigma}) \cdot \underline{\omega} \tag{Q1.1}$$

A1. Rescaling

$$\hat{\vec{\vec{v}}} = \frac{\vec{\vec{v}}}{\vec{\vec{v}} \cdot (\underline{\omega} \wedge \underline{\sigma})} \tag{A1.1}$$

then Figure A1.1 gives

$$\hat{\vec{\vec{v}}} \cdot (\underline{\omega} \wedge \underline{\sigma}) = 1 = (\hat{\vec{\vec{v}}} \cdot \underline{\sigma}) \cdot \underline{\omega} \tag{A1.2}$$

and hence Eq. (Q1.1).

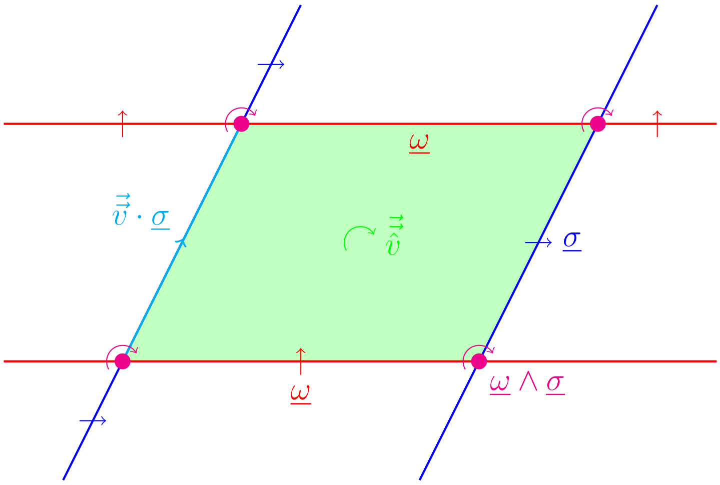

Figure A1.1: $\hat{\vec{\vec{v}}} \cdot (\underline{\omega} \wedge \underline{\sigma}) = (\hat{\vec{\vec{v}}} \cdot \underline{\sigma}) \cdot \underline{\omega}$

Q2. Consider a spacetime with time coordinate t, time covector

$$\boldsymbol{e}^t = \boldsymbol{\nabla} \wedge t \tag{Q2.1}$$

and time vector $\boldsymbol{e}_t$ satisfying

$$\boldsymbol{e}_t \cdot \boldsymbol{e}^t = 1 \tag{Q2.2}$$

Show that the time and spatial exterior derivatives

$$\dot{\boldsymbol{\omega}} \equiv \boldsymbol{e}_t \cdot (\boldsymbol{\nabla} \wedge \boldsymbol{\omega}) + \boldsymbol{\nabla} \wedge (\boldsymbol{e}_t \cdot \boldsymbol{\omega}) \tag{Q2.3}$$

$$\boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} \equiv \boldsymbol{e}_t \cdot \left(\boldsymbol{e}^t \wedge \boldsymbol{\nabla} \wedge \boldsymbol{\omega}\right) - \boldsymbol{e}^t \wedge \boldsymbol{\nabla} \wedge (\boldsymbol{e}_t \cdot \boldsymbol{\omega}) \tag{Q2.4}$$

commute and that

$$\boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} = 0 \tag{Q2.5}$$

A2. Eqs. (1.2.5) and (1.3.29) give

$$\begin{aligned}\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}^t\wedge\boldsymbol{\omega}\right) &= \left(\boldsymbol{\nabla}\wedge\boldsymbol{e}^t\right)\wedge\boldsymbol{\omega}-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega} \qquad \text{(A2.1)}\\ &= -\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega} \qquad \text{(A2.2)}\end{aligned}$$

and Eqs. (1.1.22), (Q2.2), (1.2.5), (1.2.4) and (1.3.29) give

$$\begin{aligned}&\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\\ &= \boldsymbol{\nabla}\wedge\left[\left(\boldsymbol{e}_t\cdot\boldsymbol{e}^t\right)\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right]-\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}^t\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\} \qquad \text{(A2.3)}\\ &= \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}-\left(\boldsymbol{\nabla}\wedge\boldsymbol{e}^t\right)\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]+\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right] \qquad \text{(A2.4)}\\ &= \boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right] \qquad \text{(A2.5)}\end{aligned}$$

Therefore, using Eqs. (A2.2) and (A2.5) and

$$\boldsymbol{e}_t\cdot\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)=0 \qquad \text{(A2.6)}$$

and

$$\boldsymbol{e}^t\wedge\boldsymbol{e}^t=0 \qquad \text{(A2.7)}$$

Eqs. (Q2.3) and (Q2.4) give

$$\begin{aligned}&\left(\boldsymbol{\nabla}^{(3)}\wedge\boldsymbol{\omega}\right)^{\cdot}\\ &= \boldsymbol{e}_t\cdot\left\{\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}+\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}\\ &\quad -\boldsymbol{e}_t\cdot\left\{\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\}-\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\} \qquad \text{(A2.8)}\\ &= \boldsymbol{e}_t\cdot\left\{\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}\\ &\quad +\boldsymbol{e}_t\cdot\left\{\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\}-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\} \qquad \text{(A2.9)}\\ &= \boldsymbol{\nabla}^{(3)}\wedge\dot{\boldsymbol{\omega}} \qquad \text{(A2.10)}\end{aligned}$$

and

$$\begin{aligned}&\boldsymbol{\nabla}^{(3)}\wedge\boldsymbol{\nabla}^{(3)}\wedge\boldsymbol{\omega}\\ &= \boldsymbol{e}_t\cdot\left\{\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}\\ &\quad -\boldsymbol{e}_t\cdot\left\{\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\}+\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\} \qquad \text{(A2.11)}\\ &= \boldsymbol{e}_t\cdot\left\{\boldsymbol{e}^t\wedge\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]\right\}\\ &\quad +\boldsymbol{e}_t\cdot\left[\boldsymbol{e}^t\wedge\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]+\boldsymbol{e}^t\wedge\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left\{\boldsymbol{e}_t\cdot\left[\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)\right]\right\} \qquad \text{(A2.12)}\\ &= 0 \qquad \text{(A2.13)}\end{aligned}$$

Alternatively, using Eqs. (Q2.3), (1.3.28), (1.2.4), (A2.2), (1.1.22), (Q2.2) and (Q2.4),

$$\begin{aligned}&\left(\boldsymbol{\nabla}^{(3)}\wedge\boldsymbol{\omega}\right)^{\cdot}\\ &= \boldsymbol{e}_t\cdot\left[\boldsymbol{\nabla}\wedge\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}-\boldsymbol{e}^t\wedge\dot{\boldsymbol{\omega}}\right)\right]+\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}-\boldsymbol{e}^t\wedge\dot{\boldsymbol{\omega}}\right)\right] \qquad \text{(A2.14)}\\ &= \boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\dot{\boldsymbol{\omega}}\right)+\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}_t\cdot\left(\boldsymbol{\nabla}\wedge\boldsymbol{\omega}\right)\right]-\boldsymbol{\nabla}\wedge\dot{\boldsymbol{\omega}}+\boldsymbol{\nabla}\wedge\left[\boldsymbol{e}^t\wedge\left(\boldsymbol{e}_t\cdot\dot{\boldsymbol{\omega}}\right)\right] \qquad \text{(A2.15)}\\ &= \boldsymbol{e}_t\cdot\left(\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\dot{\boldsymbol{\omega}}\right)-\boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\boldsymbol{\omega}\right)-\boldsymbol{e}^t\wedge\boldsymbol{\nabla}\wedge\left(\boldsymbol{e}_t\cdot\dot{\boldsymbol{\omega}}\right) \qquad \text{(A2.16)}\\ &= \boldsymbol{\nabla}^{(3)}\wedge\dot{\boldsymbol{\omega}} \qquad \text{(A2.17)}\end{aligned}$$

and, using Eqs. (1.2.4), (1.3.28), (A2.2), (A2.7) and (A2.17),

$$
\begin{aligned}
0 &= \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \boldsymbol{\omega} && \text{(A2.18)} \\
&= \boldsymbol{\nabla} \wedge \left(\boldsymbol{e}^t \wedge \dot{\boldsymbol{\omega}} + \boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} \right) && \text{(A2.19)} \\
&= -\boldsymbol{e}^t \wedge \boldsymbol{\nabla} \wedge \dot{\boldsymbol{\omega}} + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} && \text{(A2.20)} \\
&= -\boldsymbol{e}^t \wedge \boldsymbol{\nabla}^{(3)} \wedge \dot{\boldsymbol{\omega}} + \boldsymbol{e}^t \wedge \left(\boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} \right)^{\cdot} + \nabla^{(3)} \wedge \boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} && \text{(A2.21)} \\
&= \nabla^{(3)} \wedge \boldsymbol{\nabla}^{(3)} \wedge \boldsymbol{\omega} && \text{(A2.22)}
\end{aligned}
$$

Q3. Use abstract index notation to show that

$$
\vec{\vec{v}} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}} \right) = \left(\underline{\omega} \cdot \vec{\vec{v}} \right) \cdot \underline{\underline{\sigma}} + \left(\underline{\underline{\sigma}} \cdot \vec{\vec{v}} \right) \underline{\omega} \qquad \text{(Q3.1)}
$$

A3. Using Eqs. (2.1.11) and (2.1.13),

$$
\begin{aligned}
\vec{\vec{v}} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}} \right) &\leftrightarrow \frac{1}{2!} v^{[\mathbf{ab}]} \left(\omega_{\mathbf{a}} \sigma_{[\mathbf{bc}]} + \omega_{\mathbf{b}} \sigma_{[\mathbf{ca}]} + \omega_{\mathbf{c}} \sigma_{[\mathbf{ab}]} \right) && \text{(A3.1)} \\
&= \frac{1}{2} \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2} \omega_{\mathbf{b}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{ca}]} + \frac{1}{2!} \sigma_{[\mathbf{ab}]} v^{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A3.2)} \\
&= \frac{1}{2} \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2} \omega_{\mathbf{b}} v^{[\mathbf{ba}]} \sigma_{[\mathbf{ac}]} + \frac{1}{2!} \sigma_{[\mathbf{ab}]} v^{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A3.3)} \\
&= \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2!} \sigma_{[\mathbf{ab}]} v^{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A3.4)} \\
&\leftrightarrow \left(\underline{\omega} \cdot \vec{\vec{v}} \right) \cdot \underline{\underline{\sigma}} + \left(\underline{\underline{\sigma}} \cdot \vec{\vec{v}} \right) \underline{\omega} && \text{(A3.5)}
\end{aligned}
$$

Q4. The action functional for a scalar field can be expressed either geometrically

$$
S[\phi(x)] = \int L(\phi, \nabla\phi, x)\, \boldsymbol{\epsilon} \qquad \text{(Q4.1)}
$$

giving the geometric Euler-Lagrange equation

$$
\nabla_{\mathbf{a}} \left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}} \phi)} \right] - \frac{\partial L}{\partial \phi} = 0 \qquad \text{(Q4.2)}
$$

or in terms of coordinates

$$
S[\phi(x)] = \int L(\phi, \partial\phi, x) \sqrt{|g|}\, d^4 x \qquad \text{(Q4.3)}
$$

giving the coordinate Euler-Lagrange equation

$$
\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \text{(Q4.4)}
$$

where the Lagrangian density

$$
\mathcal{L} = \sqrt{|g|}\, L \qquad \text{(Q4.5)}
$$

(a) Show that Eqs. (Q4.2) and (Q4.4) are equivalent.

(b) Evaluate Eqs. (Q4.2) and (Q4.4) for

$$L = \frac{1}{2} g^{\mathbf{ab}} \left(\nabla_{\mathbf{a}}\phi\right)\left(\nabla_{\mathbf{b}}\phi\right) - V(\phi) \tag{Q4.6}$$

and show that the resulting equations are equivalent.

A4. (a) Eqs. (Q5.2.1) and (2.1.49) and

$$\frac{\partial L}{\partial(\partial_\alpha\phi)} = e^{\alpha}_{\mathbf{a}} \frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)} \tag{A4.1}$$

give

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right] = \frac{1}{\sqrt{|g|}}\partial_\alpha\left[\sqrt{|g|}\,\frac{\partial L}{\partial(\partial_\alpha\phi)}\right] \tag{A4.2}$$

therefore

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right] - \frac{\partial L}{\partial\phi} = \frac{1}{\sqrt{|g|}}\left\{\partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi}\right\} \tag{A4.3}$$

(b)

$$L = \frac{1}{2} g^{\mathbf{ab}} \left(\nabla_{\mathbf{a}}\phi\right)\left(\nabla_{\mathbf{b}}\phi\right) - V(\phi) \tag{A4.4}$$

therefore

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right] - \frac{\partial L}{\partial\phi} = g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}}\phi + \frac{\partial V}{\partial\phi} \tag{A4.5}$$

In components

$$\mathcal{L} = \sqrt{|g|}\left[\frac{1}{2} g^{\alpha\beta}\left(\partial_\alpha\phi\right)\left(\partial_\beta\phi\right) - V(\phi)\right] \tag{A4.6}$$

therefore, using Eqs. (Q9.3.2), (A9.3.3), (Q9.3.3) and (2.1.49),

$$\frac{1}{\sqrt{|g|}}\left\{\partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi}\right\} = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}\,g^{\alpha\beta}\partial_\beta\phi\right) + \frac{\partial V}{\partial\phi} \tag{A4.7}$$

$$= g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}}\phi + \frac{\partial V}{\partial\phi} \tag{A4.8}$$