Homework 3 - Maxwell's equations

Q3.1. Write Maxwell's equations in integral form and explain their meaning.

A3.1. Using Stokes' theorem, Eq. (1.2.11), Maxwell's equations, Eq. (1.3.1), become:

- The magnetic flux through the boundary of a volume is zero

$$\int_{\partial V} \underline{\underline{B}} = 0 \tag{A3.1.1}$$

- The electric flux through the boundary of a volume is equal to the charge within the volume

$$\int_{\partial V} \underline{\underline{D}} = \int_V \underline{\underline{\underline{\rho}}} = Q \tag{A3.1.2}$$

- The integral of the electric field around the boundary of a surface is equal to the integral of the rate of decrease of the magnetic flux density through the surface

$$\int_{\partial S} \underline{E} + \int_S \dot{\underline{\underline{B}}} = 0 \tag{A3.1.3}$$

- The integral of the magnetic field around the boundary of a surface is equal to the current, plus the integral of the rate of increase of the electric flux density, through the surface

$$\int_{\partial S} \underline{H} - \int_S \dot{\underline{\underline{D}}} = \int_S \underline{\underline{j}} = I \tag{A3.1.4}$$

Q3.2. What is the physical meaning of

(a) $\underline{\underline{\underline{J}}} \cdot \vec{v}$

(b) $\underline{E} \wedge \underline{\underline{j}}$

A3.2. (a) Using Eqs. (1.3.42) and (1.3.53),

$$\begin{aligned} \underline{\underline{\underline{J}}} \cdot \vec{v} &= \left(\underline{e}^t \wedge \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \right) \cdot \left(\vec{e}_t + \vec{v}_3 \right) & \text{(A3.2.1)} \\ &= \underline{e}^t \wedge \left(\underline{\underline{j}} \cdot \vec{v}_3 \right) + \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \cdot \vec{v}_3 & \text{(A3.2.2)} \end{aligned}$$

If $\vec{v}$ is the velocity of the charges composing the current density $\underline{\underline{\underline{J}}}$ then $\vec{v}$ will lie along the flux lines of $\underline{\underline{\underline{J}}}$ and so

$$\underline{\underline{\underline{J}}} \cdot \vec{v} = 0 \tag{A3.2.3}$$

and hence

$$\begin{aligned} \underline{\underline{j}} \cdot \vec{v}_3 &= 0 & \text{(A3.2.4)} \\ \underline{\underline{j}} &= \underline{\underline{\underline{\rho}}} \cdot \vec{v}_3 & \text{(A3.2.5)} \end{aligned}$$

Note that Eq. (A3.2.5) and the antisymmetry of $\underline{\underline{\rho}}$ imply

$$\underline{\underline{j}} \cdot \vec{v}_3 = \left(\underline{\underline{\rho}} \cdot \vec{v}_3 \right) \cdot \vec{v}_3 = 0 \tag{A3.2.6}$$

i.e. Eq. (A3.2.5) implies Eq. (A3.2.4). Thus the physical content of Eq. (A3.2.3) is that the spatial current density is equal to the charge density contracted with the spatial velocity of the charges.

(b) Using Eqs. (A3.2.5) and (1.1.22),

$$\underline{E} \wedge \underline{\underline{j}} = \underline{E} \wedge \left(\underline{\underline{\rho}} \cdot \vec{v} \right) \tag{A3.2.7}$$

$$= \left(\underline{E} \wedge \underline{\underline{\rho}} \right) \cdot \vec{v} + \left(\underline{E} \cdot \vec{v} \right) \underline{\underline{\rho}} \tag{A3.2.8}$$

$$= \left(\underline{E} \cdot \vec{v} \right) \underline{\underline{\rho}} \tag{A3.2.9}$$

Thus, comparing with Eq. (1.3.59), $\underline{E} \wedge \underline{\underline{j}}$ is the power density of the electric field $\underline{E}$ acting on the current density $\underline{\underline{j}}$.

Q3.3. Draw a three-dimensional diagram illustrating the electric flux generated by a charge.

A3.3.

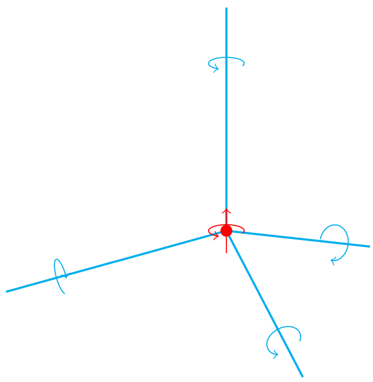

Figure A3.3.1: $\underline{\nabla} \wedge \underline{\underline{D}} = \underline{\underline{\rho}}$.

Q3.4. A current flows steadily in a straight line from A to B. Draw a two-dimensional diagram, suppressing one external dimension, illustrating the behaviour of the charge, current, electric flux and magnetic field. Explain how the orientations should be extended in the third dimension.

A3.4.

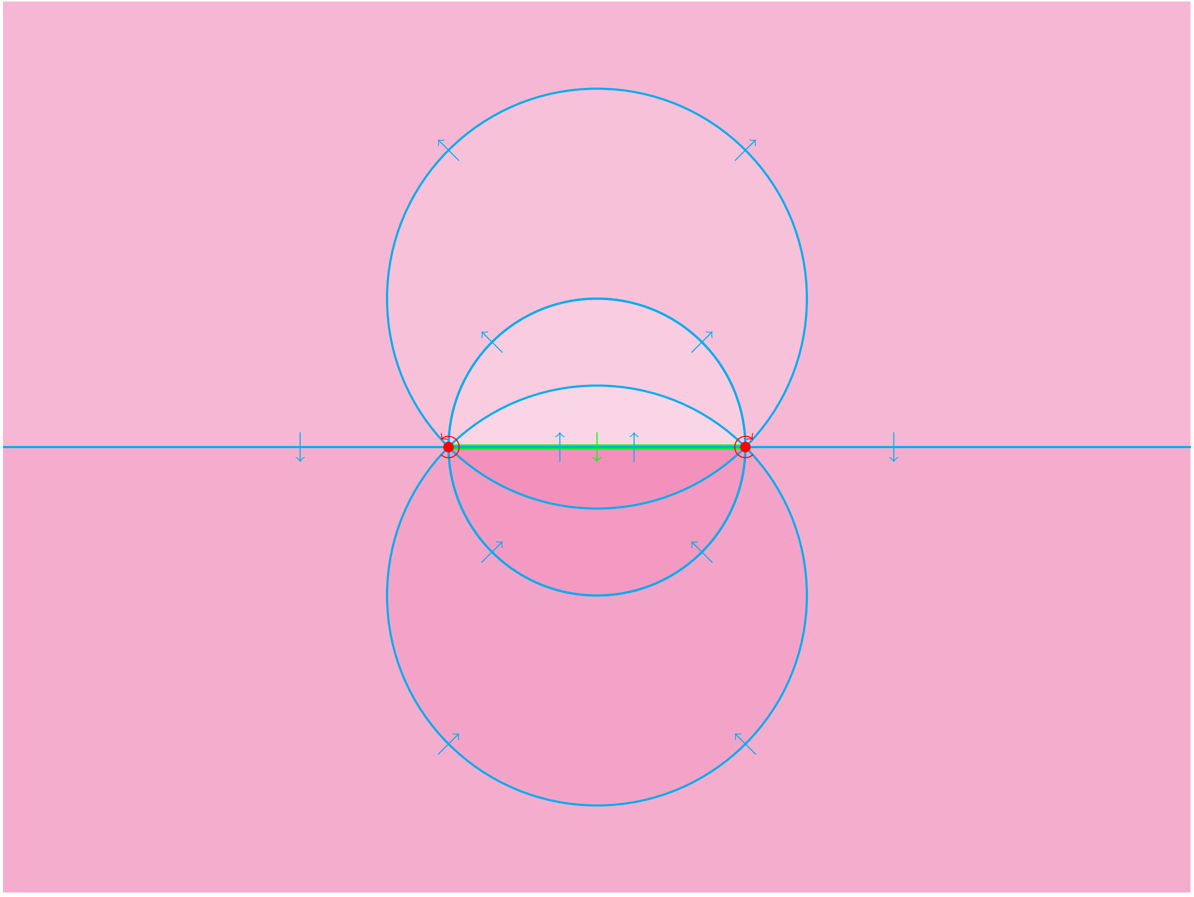

Figure A3.4.1: $\underline{\nabla} \wedge \underline{\underline{\dot{D}}} = \underline{\underline{\underline{\dot{\rho}}}} = -\underline{\nabla} \wedge \underline{\underline{j}}$ and $\underline{\nabla} \wedge \underline{H} = \underline{\underline{j}} + \underline{\underline{\dot{D}}}$ with one external dimension suppressed. The orientation arrow tips should all be extended in the same direction in the third dimension.