

Homework 4 - Densities and volumes

Q4.1. Explain the conceptual difference between $\underline{\underline{\rho}}$ and $\rho = \star \underline{\underline{\rho}}$ in three dimensions.

A4.1. $\underline{\underline{\rho}}$ is a density and $\rho = \star \underline{\underline{\rho}} = \underline{\underline{\vec{\epsilon}}} \cdot \underline{\underline{\rho}}$ combines the unit volume $\underline{\underline{\vec{\epsilon}}}$ with the density $\underline{\underline{\rho}}$ to give the traditional scalar representation of a density: amount per unit volume.

Q4.2. Show that

$$(a) \quad \underline{\underline{\epsilon}}^{-1} \cdot (\underline{\underline{v}} \cdot \underline{\underline{\epsilon}}) = \underline{\underline{v}} \quad (Q4.2.1)$$

$$(b) \quad \underline{\underline{\epsilon}}^{-1} \cdot [(\underline{\underline{u}} \wedge \underline{\underline{v}}) \cdot \underline{\underline{\epsilon}}] = \underline{\underline{u}} \wedge \underline{\underline{v}} \quad (Q4.2.2)$$

A4.2. (a) Using Eqs. (1.1.19), (1.1.22), (1.4.1) and

$$\underline{\underline{v}} \wedge \underline{\underline{\epsilon}}^{-1} = 0 \quad (A4.2.1)$$

we have

$$\underline{\underline{\epsilon}}^{-1} \cdot (\underline{\underline{v}} \cdot \underline{\underline{\epsilon}}) = -(-1)^N \underline{\underline{\epsilon}}^{-1} \cdot (\underline{\underline{\epsilon}} \cdot \underline{\underline{v}}) \quad (A4.2.2)$$

$$= -(\underline{\underline{v}} \wedge \underline{\underline{\epsilon}}^{-1}) \cdot \underline{\underline{\epsilon}} + \underline{\underline{v}} (\underline{\underline{\epsilon}}^{-1} \cdot \underline{\underline{\epsilon}}) \quad (A4.2.3)$$

$$= \underline{\underline{v}} \quad (A4.2.4)$$

(b) Similarly

$$\underline{\underline{\epsilon}}^{-1} \cdot [(\underline{\underline{u}} \wedge \underline{\underline{v}}) \cdot \underline{\underline{\epsilon}}] = \underline{\underline{\epsilon}}^{-1} \cdot [\underline{\underline{v}} \cdot (\underline{\underline{u}} \cdot \underline{\underline{\epsilon}})] \quad (A4.2.5)$$

$$= (-1)^N \underline{\underline{\epsilon}}^{-1} \cdot [(\underline{\underline{u}} \cdot \underline{\underline{\epsilon}}) \cdot \underline{\underline{v}}] \quad (A4.2.6)$$

$$= (\underline{\underline{v}} \wedge \underline{\underline{\epsilon}}^{-1}) \cdot (\underline{\underline{u}} \cdot \underline{\underline{\epsilon}}) - \underline{\underline{v}} \wedge [\underline{\underline{\epsilon}}^{-1} \cdot (\underline{\underline{u}} \cdot \underline{\underline{\epsilon}})] \quad (A4.2.7)$$

$$= -\underline{\underline{v}} \wedge \underline{\underline{u}} \quad (A4.2.8)$$

$$= \underline{\underline{u}} \wedge \underline{\underline{v}} \quad (A4.2.9)$$

Q4.3. Show that for an m -form ω and an n -vector v with $m \leq n$

$$(a) \quad (\star \omega) \cdot (\star v) = \omega \cdot v \quad (Q4.3.1)$$

$$(b) \quad \star^{-1}(\omega \wedge \star v) = v \cdot \omega \quad (Q4.3.2)$$

A4.3. (a) Using Eqs. (1.4.4) and (1.1.21), for $m + (N - n) \leq N$,

$$(\star \omega) \cdot (\star v) = (\omega \cdot \underline{\underline{\epsilon}}^{-1}) \cdot \star v = \omega \cdot (\underline{\underline{\epsilon}}^{-1} \cdot \star v) = \omega \cdot v \quad (A4.3.1)$$

(b) Using Eqs. (1.4.5), (1.1.22) and (1.4.4), for $m + (N - n) \leq N$,

$$\star^{-1}(\boldsymbol{\omega} \wedge \star \boldsymbol{v}) = \boldsymbol{\epsilon}^{-1} \cdot (\boldsymbol{\omega} \wedge \star \boldsymbol{v}) = (\boldsymbol{\epsilon}^{-1} \cdot \star \boldsymbol{v}) \cdot \boldsymbol{\omega} = \boldsymbol{v} \cdot \boldsymbol{\omega} \quad (\text{A4.3.2})$$

Q4.4. Show that

$$\nabla \cdot \nabla \cdot \boldsymbol{v} = 0 \quad (\text{Q4.4.1})$$

for any multivector \boldsymbol{v} . What traditional vector calculus results does this correspond to?

A4.4. Eq. (1.4.7) for an n -vector \boldsymbol{v} is

$$(-1)^{n-1} \nabla \cdot \boldsymbol{v} = \star^{-1} \nabla \wedge \star \boldsymbol{v} \quad (\text{A4.4.1})$$

therefore Eq. (1.2.4) gives

$$-\nabla \cdot \nabla \cdot \boldsymbol{v} = \star^{-1} \nabla \wedge \nabla \wedge \star \boldsymbol{v} = 0 \quad (\text{A4.4.2})$$

For \boldsymbol{v} a scalar or a vector, Eq. (Q4.4.1) is trivial. For \boldsymbol{v} a two-vector, Eq. (Q4.4.1) can be written as

$$0 = \underline{\nabla} \cdot \underline{\nabla} \cdot \vec{v} = -\underline{\nabla} \cdot \star^{-1} \underline{\nabla} \wedge \star \vec{v} \quad (\text{A4.4.3})$$

which corresponds to $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} = 0$ in three-dimensional vector calculus. For \boldsymbol{v} a three-vector, Eq. (Q4.4.1) can be written as

$$0 = \underline{\nabla} \cdot \underline{\nabla} \cdot \vec{\vec{v}} = -\star^{-1} \underline{\nabla} \wedge \underline{\nabla} \star \vec{\vec{v}} \quad (\text{A4.4.4})$$

which corresponds to $\vec{\nabla} \times \vec{\nabla} v = 0$ in three-dimensional vector calculus.