

## Homework 5 - Bases and coordinates

Q5.1. Write down Maxwell's equations in spherical polar coordinates.

A5.1. Using Eqs. (1.5.8) and (1.5.22), we can expand Maxwell's equations, Eq. (1.3.1), in a coordinate basis as

$$\frac{1}{2!} \frac{\partial B_{\beta\gamma}}{\partial x^\alpha} \underline{e}^\alpha \wedge \underline{e}^\beta \wedge \underline{e}^\gamma = 0 \quad (\text{A5.1.1})$$

$$\frac{\partial E_\beta}{\partial x^\alpha} \underline{e}^\alpha \wedge \underline{e}^\beta + \frac{1}{2!} \frac{\partial B_{\alpha\beta}}{\partial t} \underline{e}^\alpha \wedge \underline{e}^\beta = 0 \quad (\text{A5.1.2})$$

$$\frac{1}{2!} \frac{\partial D_{\beta\gamma}}{\partial x^\alpha} \underline{e}^\alpha \wedge \underline{e}^\beta \wedge \underline{e}^\gamma = \frac{1}{3!} \rho_{\alpha\beta\gamma} \underline{e}^\alpha \wedge \underline{e}^\beta \wedge \underline{e}^\gamma \quad (\text{A5.1.3})$$

$$\frac{\partial H_\beta}{\partial x^\alpha} \underline{e}^\alpha \wedge \underline{e}^\beta - \frac{1}{2!} \frac{\partial D_{\alpha\beta}}{\partial t} \underline{e}^\alpha \wedge \underline{e}^\beta = \frac{1}{2!} j_{\alpha\beta} \underline{e}^\alpha \wedge \underline{e}^\beta \quad (\text{A5.1.4})$$

where we have assumed that  $\dot{\underline{e}}^\alpha = 0$  as is appropriate for a spherical polar coordinate basis. The independent components are

$$\frac{1}{2!} B_{[\beta\gamma,\alpha]} = 0 \quad (\text{A5.1.5})$$

$$E_{[\beta,\alpha]} + \frac{1}{2!} \dot{B}_{[\alpha\beta]} = 0 \quad (\text{A5.1.6})$$

$$\frac{1}{2!} D_{[\beta\gamma,\alpha]} = \frac{1}{3!} \rho_{[\alpha\beta\gamma]} \quad (\text{A5.1.7})$$

$$H_{[\beta,\alpha]} - \frac{1}{2!} \dot{D}_{[\alpha\beta]} = \frac{1}{2!} j_{[\alpha\beta]} \quad (\text{A5.1.8})$$

where  $B_{\beta\gamma,\alpha} \equiv \partial B_{\beta\gamma} / \partial x^\alpha$  and square brackets denote antisymmetrization. More explicitly, in spherical polar coordinates

$$B_{\theta\phi,r} + B_{\phi r,\theta} + B_{r\theta,\phi} = 0 \quad (\text{A5.1.9})$$

$$E_{\theta,r} - E_{r,\theta} + \dot{B}_{r\theta} = 0 \quad (\text{A5.1.10})$$

$$E_{\phi,\theta} - E_{\theta,\phi} + \dot{B}_{\theta\phi} = 0 \quad (\text{A5.1.11})$$

$$E_{r,\phi} - E_{\phi,r} + \dot{B}_{\phi r} = 0 \quad (\text{A5.1.12})$$

$$D_{\theta\phi,r} + D_{\phi r,\theta} + D_{r\theta,\phi} = \rho_{r\theta\phi} \quad (\text{A5.1.13})$$

$$H_{\theta,r} - H_{r,\theta} - \dot{D}_{r\theta} = j_{r\theta} \quad (\text{A5.1.14})$$

$$H_{\phi,\theta} - H_{\theta,\phi} - \dot{D}_{\theta\phi} = j_{\theta\phi} \quad (\text{A5.1.15})$$

$$H_{r,\phi} - H_{\phi,r} - \dot{D}_{\phi r} = j_{\phi r} \quad (\text{A5.1.16})$$

Q5.2. Show that

$$\underline{\nabla} \cdot \vec{v} = \frac{1}{\epsilon_{1\dots N}} \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\alpha) \quad (\text{Q5.2.1})$$

and hence write down  $\underline{\nabla} \cdot \vec{v}$  in three dimensional spherical polar coordinates. Compare with the traditional vector calculus formula and explain the difference.

A5.2. Using Eqs. (1.4.7), (1.4.4) and (1.4.5), (1.5.2) and (1.5.13), (1.5.22), and (Q4.3.2),

$$\nabla \cdot \mathbf{v} = \star^{-1} \nabla \wedge \star \mathbf{v} \quad (\text{A5.2.1})$$

$$= \boldsymbol{\epsilon}^{-1} \cdot [\nabla \wedge (\mathbf{v} \cdot \boldsymbol{\epsilon})] \quad (\text{A5.2.2})$$

$$= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \nabla \wedge [(v^\beta \mathbf{e}_\beta) \cdot (\epsilon_{1\dots N} \mathbf{e}^1 \wedge \dots \wedge \mathbf{e}^N)] \right\} \quad (\text{A5.2.3})$$

$$= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \nabla \wedge [\epsilon_{1\dots N} v^\beta \mathbf{e}_\beta \cdot (\mathbf{e}^1 \wedge \dots \wedge \mathbf{e}^N)] \right\} \quad (\text{A5.2.4})$$

$$= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \left[ \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\beta) \right] \mathbf{e}^\alpha \wedge [\mathbf{e}_\beta \cdot (\mathbf{e}^1 \wedge \dots \wedge \mathbf{e}^N)] \right\} \quad (\text{A5.2.5})$$

$$= \left[ \frac{1}{\epsilon_{1\dots N}} \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\beta) \right] \boldsymbol{\epsilon}^{-1} \cdot [\mathbf{e}^\alpha \wedge (\mathbf{e}_\beta \cdot \boldsymbol{\epsilon})] \quad (\text{A5.2.6})$$

$$= \left[ \frac{1}{\epsilon_{1\dots N}} \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\beta) \right] \star^{-1} (\mathbf{e}^\alpha \wedge \star \mathbf{e}_\beta) \quad (\text{A5.2.7})$$

$$= \left[ \frac{1}{\epsilon_{1\dots N}} \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\beta) \right] \mathbf{e}_\beta \cdot \mathbf{e}^\alpha \quad (\text{A5.2.8})$$

$$= \frac{1}{\epsilon_{1\dots N}} \frac{\partial}{\partial x^\alpha} (\epsilon_{1\dots N} v^\alpha) \quad (\text{A5.2.9})$$

In spherical polar coordinates

$$\epsilon_{1\dots N} = r^2 \sin \theta \quad (\text{A5.2.10})$$

therefore

$$\underline{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v^r) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v^\theta) + \frac{\partial}{\partial \phi} (v^\phi) \quad (\text{A5.2.11})$$

The traditional vector calculus formula is

$$\underline{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v^{\hat{r}}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v^{\hat{\theta}}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (v^{\hat{\phi}}) \quad (\text{A5.2.12})$$

The difference arises because traditional vector calculus uses unit basis vectors  $\vec{e}_{\hat{\alpha}}$  rather than coordinate basis vectors  $\vec{e}_\alpha$

$$\vec{e}_{\hat{r}} = \vec{e}_r \quad (\text{A5.2.13})$$

$$\vec{e}_{\hat{\theta}} = \frac{1}{r} \vec{e}_\theta \quad (\text{A5.2.14})$$

$$\vec{e}_{\hat{\phi}} = \frac{1}{r \sin \theta} \vec{e}_\phi \quad (\text{A5.2.15})$$

and hence components

$$v^{\hat{r}} = v^r \quad (\text{A5.2.16})$$

$$v^{\hat{\theta}} = r v^\theta \quad (\text{A5.2.17})$$

$$v^{\hat{\phi}} = r \sin \theta v^\phi \quad (\text{A5.2.18})$$