

## Homework 6 - Abstract index notation

Q6.1. Express Eqs (1.4.3), (1.4.6) and (1.4.8) in abstract index notation.

A6.1.

$$\rho = \frac{1}{3!} \epsilon^{[abc]} \rho_{[abc]} \quad (\text{A6.1.1})$$

$$B^{\mathbf{a}} = \frac{1}{2!} \epsilon^{[abc]} B_{[bc]} \quad (\text{A6.1.2})$$

$$\nabla_{\mathbf{a}} B^{\mathbf{a}} = \frac{1}{3!} \epsilon^{[abc]} (\nabla_{\mathbf{a}} B_{[bc]} + \nabla_{\mathbf{b}} B_{[ca]} + \nabla_{\mathbf{c}} B_{[ab]}) \quad (\text{A6.1.3})$$

$$= \frac{1}{2} \epsilon^{[abc]} \nabla_{\mathbf{a}} B_{[bc]} \quad (\text{A6.1.4})$$

Q6.2. Express Eqs. (1.5.2) and (1.5.3) in abstract index notation and show that

(a) 
$$v^{\alpha} = e_{\mathbf{a}}^{\alpha} v^{\mathbf{a}} \quad (\text{Q6.2.1})$$

(b) 
$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_{\alpha} v^{\alpha} \quad (\text{Q6.2.2})$$

(c) 
$$e_{\mathbf{a}}^{\alpha} e_{\alpha}^{\mathbf{b}} = \delta_{\mathbf{a}}^{\mathbf{b}} \quad (\text{Q6.2.3})$$

explaining the meaning of all terms.

A6.2. In abstract index notation, Eq. (1.5.2) is

$$v^{\mathbf{a}} = v^{\alpha} e_{\alpha}^{\mathbf{a}} \quad (\text{A6.2.1})$$

where  $v^{\alpha}$  is the  $\alpha$  component of the vector  $v^{\mathbf{a}}$ ,  $e_{\alpha}^{\mathbf{a}}$  is the  $\alpha$  basis vector and the right hand side is summed over the components  $\alpha$ . Eq. (1.5.3) is

$$e_{\mathbf{a}}^{\alpha} e_{\beta}^{\mathbf{a}} = \delta_{\beta}^{\alpha} \quad (\text{A6.2.2})$$

where  $e_{\mathbf{a}}^{\alpha} e_{\beta}^{\mathbf{a}}$  is the  $\alpha$  basis covector contracted with the  $\beta$  basis vector and the  $\delta_{\beta}^{\alpha}$  are the components of the identity tensor  $\delta_{\mathbf{b}}^{\mathbf{a}}$ , which are equal to 1 if  $\alpha = \beta$  and zero otherwise.

(a) 
$$e_{\mathbf{a}}^{\alpha} v^{\mathbf{a}} = e_{\mathbf{a}}^{\alpha} v^{\beta} e_{\beta}^{\mathbf{a}} = v^{\beta} \delta_{\beta}^{\alpha} = v^{\alpha} \quad (\text{A6.2.3})$$

$v^{\alpha}$  is the  $\alpha$  component of the vector  $v^{\mathbf{a}}$  and  $e_{\mathbf{a}}^{\alpha} v^{\mathbf{a}}$  is the  $\alpha$  basis covector contracted with the vector  $v^{\mathbf{a}}$ .

(b)

$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_{\alpha} e_{\mathbf{a}}^{\alpha} v^{\beta} e_{\beta}^{\mathbf{a}} = \omega_{\alpha} v^{\beta} \delta_{\beta}^{\alpha} = \omega_{\alpha} v^{\alpha} \quad (\text{A6.2.4})$$

$\omega_{\mathbf{a}} v^{\mathbf{a}}$  is the covector  $\omega_{\mathbf{a}}$  contracted with the vector  $v^{\mathbf{a}}$  and  $\omega_{\alpha} v^{\alpha}$  is the sum of the products of the components of  $\omega_{\mathbf{a}}$  and  $v^{\mathbf{a}}$ .

(c)

$$e_{\mathbf{a}}^{\alpha} e_{\alpha}^{\mathbf{b}} v^{\mathbf{a}} = e_{\alpha}^{\mathbf{b}} v^{\alpha} = v^{\mathbf{b}} \quad (\text{A6.2.5})$$

therefore

$$e_{\mathbf{a}}^{\alpha} e_{\alpha}^{\mathbf{b}} = \delta_{\mathbf{a}}^{\mathbf{b}} \quad (\text{A6.2.6})$$

$e_{\mathbf{a}}^{\alpha} e_{\alpha}^{\mathbf{b}}$  is the sum of the products of the basis covectors and vectors and  $\delta_{\mathbf{a}}^{\mathbf{b}}$  is the identity tensor.

Q6.3. Using abstract index notation, show that

(a)

$$\vec{v} \cdot (\underline{\omega} \wedge \underline{\sigma}) = (\vec{v} \cdot \underline{\omega}) \underline{\sigma} + (\vec{v} \cdot \underline{\sigma}) \wedge \underline{\omega} \quad (\text{Q6.3.1})$$

(b)

$$\vec{v} \cdot (\underline{\omega} \wedge \underline{\sigma}) = \left( \vec{v} \cdot \underline{\sigma} \right) \cdot \underline{\omega} \quad (\text{Q6.3.2})$$

A6.3. (a) Using Eqs. (2.1.13) and (2.1.12),

$$\vec{v} \cdot (\underline{\omega} \wedge \underline{\sigma}) \leftrightarrow v^{\mathbf{a}} (\omega_{\mathbf{a}} \sigma_{[\mathbf{b}\mathbf{c}]} + \omega_{\mathbf{b}} \sigma_{[\mathbf{c}\mathbf{a}]} + \omega_{\mathbf{c}} \sigma_{[\mathbf{a}\mathbf{b}]}) \quad (\text{A6.3.1})$$

$$= v^{\mathbf{a}} \omega_{\mathbf{a}} \sigma_{[\mathbf{b}\mathbf{c}]} + v^{\mathbf{a}} \sigma_{[\mathbf{a}\mathbf{b}]} \omega_{\mathbf{c}} + v^{\mathbf{a}} \sigma_{[\mathbf{c}\mathbf{a}]} \omega_{\mathbf{b}} \quad (\text{A6.3.2})$$

$$= v^{\mathbf{a}} \omega_{\mathbf{a}} \sigma_{[\mathbf{b}\mathbf{c}]} + v^{\mathbf{a}} \sigma_{[\mathbf{a}\mathbf{b}]} \omega_{\mathbf{c}} - v^{\mathbf{a}} \sigma_{[\mathbf{a}\mathbf{c}]} \omega_{\mathbf{b}} \quad (\text{A6.3.3})$$

$$\leftrightarrow (\vec{v} \cdot \underline{\omega}) \underline{\sigma} + (\vec{v} \cdot \underline{\sigma}) \wedge \underline{\omega} \quad (\text{A6.3.4})$$

(b) Using Eqs. (2.1.10) and (2.1.12),

$$\vec{v} \cdot (\underline{\omega} \wedge \underline{\sigma}) \leftrightarrow \frac{1}{2!} v^{[\mathbf{a}\mathbf{b}]} (\omega_{\mathbf{a}} \sigma_{\mathbf{b}} - \omega_{\mathbf{b}} \sigma_{\mathbf{a}}) \quad (\text{A6.3.5})$$

$$= \frac{1}{2} v^{[\mathbf{a}\mathbf{b}]} \sigma_{\mathbf{b}} \omega_{\mathbf{a}} - \frac{1}{2} v^{[\mathbf{a}\mathbf{b}]} \sigma_{\mathbf{a}} \omega_{\mathbf{b}} \quad (\text{A6.3.6})$$

$$= \frac{1}{2} v^{[\mathbf{a}\mathbf{b}]} \sigma_{\mathbf{b}} \omega_{\mathbf{a}} + \frac{1}{2} v^{[\mathbf{b}\mathbf{a}]} \sigma_{\mathbf{a}} \omega_{\mathbf{b}} \quad (\text{A6.3.7})$$

$$= v^{[\mathbf{a}\mathbf{b}]} \sigma_{\mathbf{b}} \omega_{\mathbf{a}} \quad (\text{A6.3.8})$$

$$\leftrightarrow \left( \vec{v} \cdot \underline{\sigma} \right) \cdot \underline{\omega} \quad (\text{A6.3.9})$$