

Homework 7 - Metric

Q7.1. Let \vec{e}_r and \vec{e}_θ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ be the orthonormal basis vectors proportional to \vec{e}_r and \vec{e}_θ .

- Express the metric $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases.
- Express the coordinate basis vectors and covectors in terms of the orthonormal basis vectors and covectors.
- Draw simple diagrams illustrating \vec{e}_r , \vec{e}_θ , \underline{e}^r , \underline{e}^θ , $\vec{e}_{\hat{r}}$, $\vec{e}_{\hat{\theta}}$, $\underline{e}^{\hat{r}}$ and $\underline{e}^{\hat{\theta}}$.
- Express the volume form $\epsilon_{\mathbf{ab}}$ and volume element $\epsilon^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases. Draw simple diagrams illustrating the origin of the extra factors in the case of the coordinate basis.

A7.1. (a) In polar coordinates

$$g_{\alpha\beta} dx^\alpha dx^\beta = ds^2 = dr^2 + r^2 d\theta^2 \quad (\text{A7.1.1})$$

therefore $g_{rr} = 1$, $g_{r\theta} = 0$ and $g_{\theta\theta} = r^2$, and using $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$ gives $g^{rr} = 1$, $g^{r\theta} = 0$ and $g^{\theta\theta} = r^{-2}$. Therefore

$$g_{\mathbf{ab}} = g_{\alpha\beta} e_{\mathbf{a}}^\alpha e_{\mathbf{b}}^\beta = e_{\mathbf{a}}^r e_{\mathbf{b}}^r + r^2 e_{\mathbf{a}}^\theta e_{\mathbf{b}}^\theta \quad (\text{A7.1.2})$$

and

$$g^{\mathbf{ab}} = g^{\alpha\beta} e_{\alpha}^{\mathbf{a}} e_{\beta}^{\mathbf{b}} = e_r^{\mathbf{a}} e_r^{\mathbf{b}} + \frac{1}{r^2} e_{\theta}^{\mathbf{a}} e_{\theta}^{\mathbf{b}} \quad (\text{A7.1.3})$$

In an orthonormal basis

$$g_{\alpha\beta} = g_{\mathbf{ab}} e_{\alpha}^{\mathbf{a}} e_{\beta}^{\mathbf{b}} = \delta_{\alpha\beta} \quad (\text{A7.1.4})$$

therefore

$$g_{\mathbf{ab}} = e_{\mathbf{a}}^{\hat{r}} e_{\mathbf{b}}^{\hat{r}} + e_{\mathbf{a}}^{\hat{\theta}} e_{\mathbf{b}}^{\hat{\theta}} \quad (\text{A7.1.5})$$

and

$$g^{\mathbf{ab}} = e_{\hat{r}}^{\mathbf{a}} e_{\hat{r}}^{\mathbf{b}} + e_{\hat{\theta}}^{\mathbf{a}} e_{\hat{\theta}}^{\mathbf{b}} \quad (\text{A7.1.6})$$

(b) Comparing Eqs. (A7.1.2) and (A7.1.5) and Eqs. (A7.1.3) and (A7.1.6) gives

$$\underline{e}^r = \underline{e}^{\hat{r}} \quad , \quad \underline{e}^\theta = \frac{1}{r} \underline{e}^{\hat{\theta}} \quad (\text{A7.1.7})$$

$$\vec{e}_r = \vec{e}_{\hat{r}} \quad , \quad \vec{e}_\theta = r \vec{e}_{\hat{\theta}} \quad (\text{A7.1.8})$$

(c) See Figure A7.1.1.

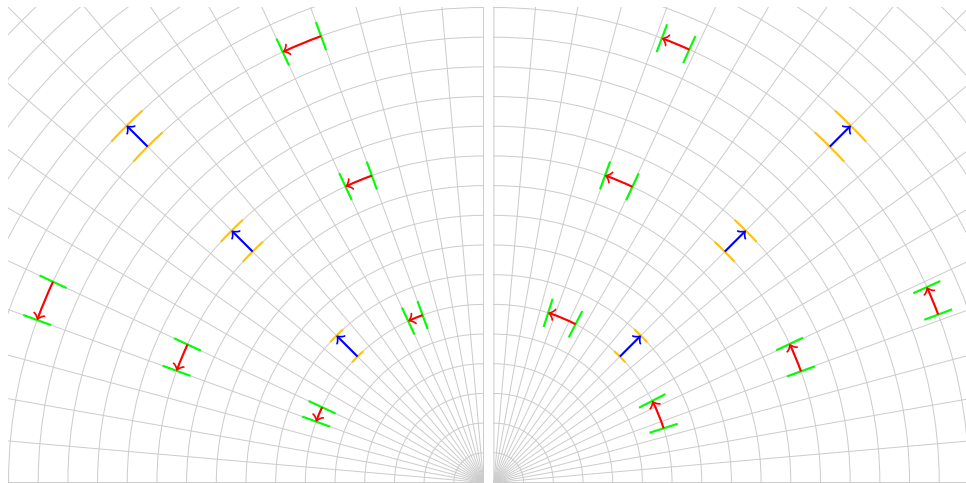


Figure A7.1.1: Left: polar coordinate basis, $\vec{e}_r, \vec{e}_\theta, \underline{e}^r, \underline{e}^\theta$. Right: polar orthonormal basis, $\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \underline{e}^{\hat{r}}, \underline{e}^{\hat{\theta}}$. Comparing: $\vec{e}_r = \vec{e}_{\hat{r}}, \underline{e}^r = \underline{e}^{\hat{r}}, \vec{e}_\theta = r\vec{e}_{\hat{\theta}}, \underline{e}^\theta = r^{-1}\underline{e}^{\hat{\theta}}$.

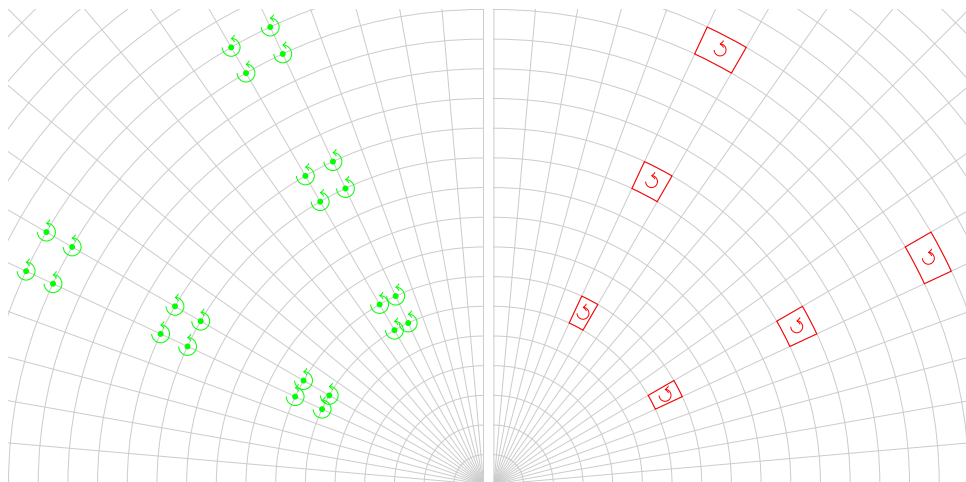


Figure A7.1.2: Polar coordinate volume form $\underline{e}^r \wedge \underline{e}^\theta = r^{-1}\underline{\epsilon}$ and element $\vec{e}_r \wedge \vec{e}_\theta = r\vec{\epsilon}$.

(d)

$$\underline{\epsilon} = \epsilon_{12} \underline{e}^1 \wedge \underline{e}^2 = r \underline{e}^r \wedge \underline{e}^\theta \tag{A7.1.9}$$

$$= \underline{e}^{\hat{r}} \wedge \underline{e}^{\hat{\theta}} \tag{A7.1.10}$$

The polar coordinate volume form is inversely proportional to r , see Figure A7.1.2.

$$\vec{\epsilon} = \frac{1}{\epsilon_{12}} \vec{e}_1 \wedge \vec{e}_2 = \frac{1}{r} \vec{e}_r \wedge \vec{e}_\theta \tag{A7.1.11}$$

$$= \vec{e}_{\hat{r}} \wedge \vec{e}_{\hat{\theta}} \tag{A7.1.12}$$

The polar coordinate volume element is proportional to r , see Figure A7.1.2.

Q7.2. What is the geometrical meaning of $(\vec{u} \cdot \vec{v}) \cdot (\vec{u} \cdot \vec{v})$?

A7.2. Decomposing¹

$$\vec{v} = \vec{s} \wedge \vec{t} \quad (\text{A7.2.1})$$

with

$$\vec{t} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} \quad (\text{A7.2.2})$$

$$\vec{s} = \frac{\vec{v} \cdot \vec{t}}{|\vec{t}|^2} \vec{t} \quad (\text{A7.2.3})$$

so that

$$\vec{t} \cdot \vec{u} = 0 \quad (\text{A7.2.4})$$

$$\vec{s} \cdot \vec{t} = 0 \quad (\text{A7.2.5})$$

$$\vec{s} \cdot \vec{u} = 1 \quad (\text{A7.2.6})$$

gives

$$(\vec{u} \cdot \vec{v}) \cdot (\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \vec{s})^2 (\vec{t} \cdot \vec{t}) \quad (\text{A7.2.7})$$

$$= |\vec{u}|^2 |\vec{s}|^2 |\vec{t}|^2 \cos^2 \theta \quad (\text{A7.2.8})$$

$$= |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \quad (\text{A7.2.9})$$

where θ is the angle between \vec{u} and \vec{v} , see Figure A7.2.1.

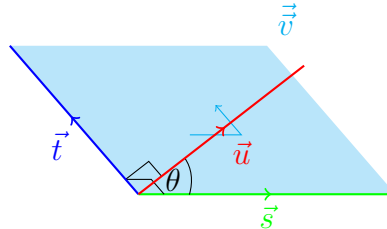


Figure A7.2.1: $\vec{u} \cdot \vec{v} = \vec{u} \cdot (\vec{s} \wedge \vec{t}) = (\vec{u} \cdot \vec{s}) \vec{t} = (|\vec{u}| |\vec{s}| \cos \theta) \vec{t}$.

Q7.3. Express the traditional vector calculus curl of a vector field in terms of the exterior derivative, and hence show that

$$\vec{\nabla} \times \vec{v} = \frac{\vec{e}_\alpha}{\epsilon_{\alpha\beta\gamma}} \frac{\partial}{\partial x^\beta} (g_{\gamma\delta} v^\delta) \quad (\text{Q7.3.1})$$

Hence derive the textbook formula for $\vec{\nabla} \times \vec{v}$ in spherical polar coordinates. Compare with the formula for $\underline{\nabla} \wedge \underline{v}$ in spherical polar coordinates.

¹In four or more dimensions, it may only be possible to decompose \vec{v} into a sum of such terms from independent planes.

A7.3. Using Eqs. (1.5.22), (1.4.4) and (1.5.15),

$$\vec{\nabla} \times \vec{v} = \star \underline{\nabla} \wedge \underline{v} \quad (\text{A7.3.1})$$

$$= \star \underline{\nabla} \wedge (\underline{e}^\gamma g_{\gamma\delta} v^\delta) \quad (\text{A7.3.2})$$

$$= \star (\underline{e}^\beta \wedge \underline{e}^\gamma) \frac{\partial}{\partial x^\beta} (g_{\gamma\delta} v^\delta) \quad (\text{A7.3.3})$$

$$= \frac{\vec{e}_\alpha}{\epsilon_{\alpha\beta\gamma}} \frac{\partial}{\partial x^\beta} (g_{\gamma\delta} v^\delta) \quad (\text{A7.3.4})$$

In three dimensional polar coordinates, the above becomes

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \frac{1}{\epsilon_{r\theta\phi}} \left[\frac{\partial}{\partial \theta} (g_{\phi\phi} v^\phi) - \frac{\partial}{\partial \phi} (g_{\theta\theta} v^\theta) \right] \vec{e}_r \\ &+ \frac{1}{\epsilon_{r\theta\phi}} \left[\frac{\partial}{\partial \phi} (g_{rr} v^r) - \frac{\partial}{\partial r} (g_{\phi\phi} v^\phi) \right] \vec{e}_\theta \\ &+ \frac{1}{\epsilon_{r\theta\phi}} \left[\frac{\partial}{\partial r} (g_{\theta\theta} v^\theta) - \frac{\partial}{\partial \theta} (g_{rr} v^r) \right] \vec{e}_\phi \end{aligned} \quad (\text{A7.3.5})$$

Now $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2 \theta$ and $\epsilon_{r\theta\phi} = r^2 \sin \theta$, therefore

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \frac{1}{\sin \theta} \left[\frac{\partial}{\partial \theta} (\sin^2 \theta v^\phi) - \frac{\partial v^\theta}{\partial \phi} \right] \vec{e}_r \\ &+ \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial v^r}{\partial \phi} - \frac{\partial}{\partial r} (r^2 \sin \theta v^\phi) \right] \vec{e}_\theta \\ &+ \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 v^\theta) - \frac{\partial v^r}{\partial \theta} \right] \vec{e}_\phi \end{aligned} \quad (\text{A7.3.6})$$

The textbook formula uses the orthonormal basis

$$\vec{e}_{\hat{r}} = \vec{e}_r \quad , \quad \vec{e}_{\hat{\theta}} = \frac{1}{r} \vec{e}_\theta \quad , \quad \vec{e}_{\hat{\phi}} = \frac{1}{r \sin \theta} \vec{e}_\phi \quad (\text{A7.3.7})$$

with components

$$v^{\hat{r}} = v^r \quad , \quad v^{\hat{\theta}} = r v^\theta \quad , \quad v^{\hat{\phi}} = r \sin \theta v^\phi \quad (\text{A7.3.8})$$

Therefore the textbook formula is

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v^{\hat{\phi}}) - \frac{\partial v^{\hat{\theta}}}{\partial \phi} \right] \vec{e}_{\hat{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v^{\hat{r}}}{\partial \phi} - \frac{\partial}{\partial r} (r v^{\hat{\phi}}) \right] \vec{e}_{\hat{\theta}} \\ &+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r v^{\hat{\theta}}) - \frac{\partial v^{\hat{r}}}{\partial \theta} \right] \vec{e}_{\hat{\phi}} \end{aligned} \quad (\text{A7.3.9})$$

In comparison

$$\underline{\nabla} \wedge \underline{v} = \left(\frac{\partial v_\phi}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right) \underline{e}^\theta \wedge \underline{e}^\phi + \left(\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} \right) \underline{e}^\phi \wedge \underline{e}^r + \left(\frac{\partial v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \underline{e}^r \wedge \underline{e}^\theta \quad (\text{A7.3.10})$$