

Homework 10 - Christoffel symbols

Q10.1. Show that

$$\nabla_{\mathbf{a}} e_{\mathbf{b}}^{\gamma} = -\Gamma_{\alpha\beta}^{\gamma} e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} \quad (\text{Q10.1.1})$$

and hence derive Eq. (2.2.9) and show that, for zero torsion and in a coordinate basis,

$$\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma} \quad (\text{Q10.1.2})$$

A10.1.

$$0 = \nabla_{\mathbf{a}} \delta_{\beta}^{\gamma} = \nabla_{\mathbf{a}} (e_{\beta}^{\mathbf{c}} e_{\mathbf{c}}^{\gamma}) = (\nabla_{\mathbf{a}} e_{\beta}^{\mathbf{c}}) e_{\mathbf{c}}^{\gamma} + e_{\beta}^{\mathbf{c}} (\nabla_{\mathbf{a}} e_{\mathbf{c}}^{\gamma}) \quad (\text{A10.1.1})$$

therefore

$$\nabla_{\mathbf{a}} e_{\mathbf{b}}^{\gamma} = -e_{\mathbf{b}}^{\beta} (\nabla_{\mathbf{a}} e_{\beta}^{\mathbf{c}}) e_{\mathbf{c}}^{\gamma} = -e_{\mathbf{b}}^{\beta} \Gamma_{\mathbf{a}\beta}^{\mathbf{c}} e_{\mathbf{c}}^{\gamma} = -\Gamma_{\alpha\beta}^{\gamma} e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} \quad (\text{A10.1.2})$$

and hence

$$\nabla_{\mathbf{a}} \omega_{\mathbf{b}} = \nabla_{\mathbf{a}} (\omega_{\beta} e_{\mathbf{b}}^{\beta}) = (\nabla_{\mathbf{a}} \omega_{\beta}) e_{\mathbf{b}}^{\beta} + \omega_{\beta} \nabla_{\mathbf{a}} e_{\mathbf{b}}^{\beta} = \left(\frac{\partial \omega_{\beta}}{\partial x^{\alpha}} - \omega_{\gamma} \Gamma_{\alpha\beta}^{\gamma} \right) e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} \quad (\text{A10.1.3})$$

For zero torsion and in a coordinate basis,

$$\nabla_{\mathbf{a}} e_{\mathbf{b}}^{\gamma} = \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} x^{\gamma} = \nabla_{\mathbf{b}} \nabla_{\mathbf{a}} x^{\gamma} = \nabla_{\mathbf{b}} e_{\mathbf{a}}^{\gamma} \quad (\text{A10.1.4})$$

therefore from Eq. (A10.1.2)

$$\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma} \quad (\text{A10.1.5})$$

Q10.2. Derive Eq. (2.2.10).

A10.2. Eqs. (2.2.2) and (Q10.1.1) give

$$0 = \nabla_{\mathbf{a}} g_{\mathbf{bc}} \quad (\text{A10.2.1})$$

$$= \nabla_{\mathbf{a}} (g_{\beta\gamma} e_{\mathbf{b}}^{\beta} e_{\mathbf{c}}^{\gamma}) \quad (\text{A10.2.2})$$

$$= g_{\beta\gamma,\alpha} e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} e_{\mathbf{c}}^{\gamma} - g_{\beta\gamma} \Gamma_{\mathbf{ab}}^{\beta} e_{\mathbf{c}}^{\gamma} - g_{\beta\gamma} e_{\mathbf{b}}^{\beta} \Gamma_{\mathbf{ac}}^{\gamma} \quad (\text{A10.2.3})$$

where subscript α denotes the partial derivative with respect to x^{α} . Therefore

$$g_{\beta\gamma,\alpha} = g_{\delta\gamma} \Gamma_{\alpha\beta}^{\delta} + g_{\beta\delta} \Gamma_{\alpha\gamma}^{\delta} \quad (\text{A10.2.4})$$

and

$$\begin{aligned} & g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} \\ &= g_{\delta\beta} \Gamma_{\gamma\alpha}^{\delta} + g_{\alpha\delta} \Gamma_{\gamma\beta}^{\delta} + g_{\delta\gamma} \Gamma_{\beta\alpha}^{\delta} + g_{\alpha\delta} \Gamma_{\beta\gamma}^{\delta} - g_{\delta\gamma} \Gamma_{\alpha\beta}^{\delta} - g_{\beta\delta} \Gamma_{\alpha\gamma}^{\delta} \end{aligned} \quad (\text{A10.2.5})$$

$$= g_{\alpha\delta} (\Gamma_{\gamma\beta}^{\delta} + \Gamma_{\beta\gamma}^{\delta}) + g_{\delta\gamma} (\Gamma_{\beta\alpha}^{\delta} - \Gamma_{\alpha\beta}^{\delta}) + g_{\beta\delta} (\Gamma_{\gamma\alpha}^{\delta} - \Gamma_{\alpha\gamma}^{\delta}) \quad (\text{A10.2.6})$$

Therefore, using Eq. (Q10.1.2)

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) \quad (\text{A10.2.7})$$

Q10.3. Express the curvature tensor in terms of the Christoffel symbols.

A10.3. Using Eqs. (2.2.4) and (Q10.1.1),

$$R_{\mathbf{abc}}^{\mathbf{e}} e_{\mathbf{e}}^{\delta} = (\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}) e_{\mathbf{c}}^{\delta} = -\nabla_{\mathbf{a}} \Gamma_{\mathbf{bc}}^{\delta} + \nabla_{\mathbf{b}} \Gamma_{\mathbf{ac}}^{\delta} \quad (\text{A10.3.1})$$

Therefore, using Eq. (2.2.7),

$$= -\nabla_{\mathbf{a}} \Gamma_{\mathbf{bc}}^{\mathbf{d}} + \Gamma_{\mathbf{ae}}^{\mathbf{d}} \Gamma_{\mathbf{bc}}^{\mathbf{e}} + \nabla_{\mathbf{b}} \Gamma_{\mathbf{ac}}^{\mathbf{d}} - \Gamma_{\mathbf{be}}^{\mathbf{d}} \Gamma_{\mathbf{ac}}^{\mathbf{e}} \quad (\text{A10.3.2})$$

$$= -\nabla_{\mathbf{a}} \Gamma_{\mathbf{bc}}^{\mathbf{d}} + \Gamma_{\mathbf{bc}}^{\delta} \nabla_{\mathbf{a}} e_{\delta}^{\mathbf{d}} + \nabla_{\mathbf{b}} \Gamma_{\mathbf{ac}}^{\mathbf{d}} - \Gamma_{\mathbf{ac}}^{\delta} \nabla_{\mathbf{b}} e_{\delta}^{\mathbf{d}} \quad (\text{A10.3.3})$$

$$R_{\mathbf{abc}}^{\mathbf{d}} = -e_{\delta}^{\mathbf{d}} \nabla_{\mathbf{a}} \Gamma_{\mathbf{bc}}^{\delta} + e_{\delta}^{\mathbf{d}} \nabla_{\mathbf{b}} \Gamma_{\mathbf{ac}}^{\delta} \quad (\text{A10.3.4})$$

$$= -e_{\delta}^{\mathbf{d}} \nabla_{\mathbf{a}} \left(\Gamma_{\beta\gamma}^{\delta} e_{\mathbf{b}}^{\beta} e_{\mathbf{c}}^{\gamma} \right) + e_{\delta}^{\mathbf{d}} \nabla_{\mathbf{b}} \left(\Gamma_{\alpha\gamma}^{\delta} e_{\mathbf{a}}^{\alpha} e_{\mathbf{c}}^{\gamma} \right) \quad (\text{A10.3.5})$$

$$= \left(-\nabla_{\alpha} \Gamma_{\beta\gamma}^{\delta} + \Gamma_{\epsilon\gamma}^{\delta} \Gamma_{\alpha\beta}^{\epsilon} + \Gamma_{\beta\epsilon}^{\delta} \Gamma_{\alpha\gamma}^{\epsilon} + \nabla_{\beta} \Gamma_{\alpha\gamma}^{\delta} - \Gamma_{\epsilon\gamma}^{\delta} \Gamma_{\beta\alpha}^{\epsilon} - \Gamma_{\alpha\epsilon}^{\delta} \Gamma_{\beta\gamma}^{\epsilon} \right) \\ \times e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} e_{\mathbf{c}}^{\gamma} e_{\delta}^{\mathbf{d}} \quad (\text{A10.3.6})$$

$$= \left(-\nabla_{\alpha} \Gamma_{\beta\gamma}^{\delta} - \Gamma_{\alpha\epsilon}^{\delta} \Gamma_{\beta\gamma}^{\epsilon} + \nabla_{\beta} \Gamma_{\alpha\gamma}^{\delta} + \Gamma_{\beta\epsilon}^{\delta} \Gamma_{\alpha\gamma}^{\epsilon} \right) e_{\mathbf{a}}^{\alpha} e_{\mathbf{b}}^{\beta} e_{\mathbf{c}}^{\gamma} e_{\delta}^{\mathbf{d}} \quad (\text{A10.3.7})$$

where Eq. (A10.3.7) assumes a coordinate basis and uses Eq. (Q10.1.2).

Q10.4. Let $e_r^{\mathbf{a}}$ and $e_{\theta}^{\mathbf{a}}$ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $e_{\hat{r}}^{\mathbf{a}}$ and $e_{\hat{\theta}}^{\mathbf{a}}$ be the orthonormal basis vectors proportional to $e_r^{\mathbf{a}}$ and $e_{\theta}^{\mathbf{a}}$.

- Calculate the $\Gamma_{\alpha\beta}^{\gamma}$ for the coordinate basis.
- Express the velocity and acceleration in terms of the coordinate and orthonormal bases.
- Write down the equation of a geodesic in the coordinate basis and check that the geodesics are straight lines.

A10.4. (a) Using Eqs. (A10.2.7) and (2.1.33), the only non-zero derivative of the metric components is

$$\frac{\partial g_{\theta\theta}}{\partial r} = 2r \quad (\text{A10.4.1})$$

therefore

$$\Gamma_{\theta\theta}^r = -r \quad , \quad \Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \frac{1}{r} \quad (\text{A10.4.2})$$

and others zero.

(b) Eqs. (2.2.11) and (A6.1.8) give

$$v^{\mathbf{a}} = \frac{dx^{\alpha}}{dt} e_{\alpha}^{\mathbf{a}} = \dot{r} e_r^{\mathbf{a}} + \dot{\theta} e_{\theta}^{\mathbf{a}} \quad (\text{A10.4.3})$$

$$= \dot{r} e_{\hat{r}}^{\mathbf{a}} + r \dot{\theta} e_{\hat{\theta}}^{\mathbf{a}} \quad (\text{A10.4.4})$$

and Eqs. (2.2.12) and (A10.4.2) give

$$a^{\mathbf{a}} = \left(\frac{d^2 x^\alpha}{dt^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \right) e_\alpha^{\mathbf{a}} \quad (\text{A10.4.5})$$

$$= \left(\ddot{r} - r\dot{\theta}^2 \right) e_r^{\mathbf{a}} + \left(\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} \right) e_\theta^{\mathbf{a}} \quad (\text{A10.4.6})$$

$$= \left(\ddot{r} - r\dot{\theta}^2 \right) e_{\hat{r}}^{\mathbf{a}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) e_{\hat{\theta}}^{\mathbf{a}} \quad (\text{A10.4.7})$$

(c) The geodesic equation is

$$a^{\mathbf{a}} = 0 \quad (\text{A10.4.8})$$

therefore

$$\ddot{r} - r\dot{\theta}^2 = 0 \quad (\text{A10.4.9})$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad (\text{A10.4.10})$$

corresponding to straight lines, though not obviously so.