Homework 11 - Calculus of variations

Q11.1. The action functional

$$S[x(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(x, \dot{x}, t)\, dt \tag{Q11.1.1}$$

can be varied either covariantly

$$\frac{\delta S}{\delta x^{\mathbf{a}}} = \frac{\partial L}{\partial x^{\mathbf{a}}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) \tag{Q11.1.2}$$

or with respect to the coordinate paths

$$\frac{\delta S}{\delta x^{\alpha}} = \frac{\partial L}{\partial x^{\alpha}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) \tag{Q11.1.3}$$

(a) Show that

$$\frac{\delta S}{\delta x^{\mathbf{a}}} = \frac{\delta S}{\delta x^{\alpha}} e^{\alpha}_{\mathbf{a}} \tag{Q11.1.4}$$

(b) Evaluate Eqs. (Q11.1.2) and (Q11.1.3) for

$$L = \frac{1}{2} m g_{\mathbf{ab}} \dot{x}^{\mathbf{a}} \dot{x}^{\mathbf{b}} - V(x) \tag{Q11.1.5}$$

and show that they are equivalent.

A11.1. (a)

$$\begin{aligned} \frac{\delta S}{\delta x^{\mathbf{a}}} &= \frac{\partial L}{\partial x^{\mathbf{a}}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) \\ &= \frac{\partial L}{\partial x^{\alpha}} e^{\alpha}_{\mathbf{a}} + \frac{\partial L}{\partial \dot{x}^{\alpha}} \dot{e}^{\alpha}_{\mathbf{a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}} e^{\alpha}_{\mathbf{a}}\right) \\ &= \left[\frac{\partial L}{\partial x^{\alpha}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right)\right] e^{\alpha}_{\mathbf{a}} \\ &= \frac{\delta S}{\delta x^{\alpha}} e^{\alpha}_{\mathbf{a}} \end{aligned} \tag{A11.1.1}$$

(b)

$$L = \frac{1}{2} m g_{\mathbf{ab}} \dot{x}^{\mathbf{a}} \dot{x}^{\mathbf{b}} - V(x) \tag{A11.1.2}$$

therefore

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) - \frac{\partial L}{\partial x^{\mathbf{a}}} &= \frac{d}{dt}\left(m g_{\mathbf{ab}} \dot{x}^{\mathbf{b}}\right) + \frac{\partial V}{\partial x^{\mathbf{a}}} \\ &= m g_{\mathbf{ab}} \ddot{x}^{\mathbf{b}} + \nabla_{\mathbf{a}} V \end{aligned} \tag{A11.1.3}$$

In components

$$L = \frac{1}{2} m g_{\gamma\delta} \dot{x}^{\gamma} \dot{x}^{\delta} - V(x) \tag{A11.1.4}$$

therefore

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} &= \frac{d}{dt}\left(m g_{\alpha\beta}\dot{x}^\beta\right) - \frac{1}{2}m\frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m\dot{x}^\epsilon\frac{\partial g_{\alpha\beta}}{\partial x^\epsilon}\dot{x}^\beta + m g_{\alpha\beta}\ddot{x}^\beta - \frac{1}{2}m\frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m g_{\alpha\beta}\ddot{x}^\beta + \frac{1}{2}m\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\delta} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\right)\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m g_{\alpha\beta}\left(\ddot{x}^\beta + \Gamma^\beta_{\gamma\delta}\dot{x}^\gamma\dot{x}^\delta\right) + \frac{\partial V}{\partial x^\alpha} \qquad \text{(A11.1.5)}
\end{aligned}
$$

which is the component form of Eq. (A11.1.3).