Homework 12 - Symmetry

Q12.1. (a) Express $\mathcal{L}_u v^{\mathbf{a}}$ in a coordinate basis and deduce that it is independent of the metric.

(b) Use Eq. (2.3.20) to show that

$$\mathcal{L}_u \omega_{\mathbf{a}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \qquad \text{(Q12.1.1)}$$

and check that this is consistent with Eq. (1.2.6).

Q12.2. (a) Derive Eq. (2.3.21).

(b) Show that a coordinate basis vector $e_\alpha^{\mathbf{a}}$ is a Killing vector if and only if

$$\nabla_\alpha g_{\beta\gamma} = 0 \qquad \text{(Q12.2.1)}$$

for all β, γ, and explain the difference between $\nabla_\alpha g_{\beta\gamma}$ and $\nabla_{\mathbf{a}} g_{\mathbf{bc}}$.

(c) Show that a particle with momentum

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \qquad \text{(Q12.2.2)}$$

and moving freely in a space with Killing vector field $\xi^{\mathbf{a}}$ has conserved quantity $\xi^{\mathbf{a}} p_{\mathbf{a}}$.