

Homework 12 - Symmetry

- Q12.1. (a) Express $\mathcal{L}_u v^{\mathbf{a}}$ in a coordinate basis and deduce that it is independent of the metric.
 (b) Use Eq. (2.3.20) to show that

$$\mathcal{L}_u \omega_{\mathbf{a}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \quad (\text{Q12.1.1})$$

and check that this is consistent with Eq. (1.2.6).

- A12.1. (a) Using Eqs. (2.3.20), (2.2.8) and (Q10.1.2), in a coordinate basis,

$$\mathcal{L}_u v^{\mathbf{a}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} \quad (\text{A12.1.1})$$

$$= u^{\beta} \left(\frac{\partial v^{\alpha}}{\partial x^{\beta}} e_{\alpha}^{\mathbf{a}} + v^{\alpha} \Gamma_{\beta\alpha}^{\gamma} e_{\gamma}^{\mathbf{a}} \right) - v^{\beta} \left(\frac{\partial u^{\alpha}}{\partial x^{\beta}} e_{\alpha}^{\mathbf{a}} + u^{\alpha} \Gamma_{\beta\alpha}^{\gamma} e_{\gamma}^{\mathbf{a}} \right) \quad (\text{A12.1.2})$$

$$= \left(u^{\beta} \frac{\partial v^{\alpha}}{\partial x^{\beta}} - v^{\beta} \frac{\partial u^{\alpha}}{\partial x^{\beta}} \right) e_{\alpha}^{\mathbf{a}} \quad (\text{A12.1.3})$$

which is independent of the metric.

- (b) Using the Leibniz rule and Eq. (2.3.20),

$$v^{\mathbf{a}} \mathcal{L}_u \omega_{\mathbf{a}} = \mathcal{L}_u (\omega_{\mathbf{a}} v^{\mathbf{a}}) - \omega_{\mathbf{a}} \mathcal{L}_u v^{\mathbf{a}} \quad (\text{A12.1.4})$$

$$= u^{\mathbf{b}} \nabla_{\mathbf{b}} (\omega_{\mathbf{a}} v^{\mathbf{a}}) - \omega_{\mathbf{a}} (u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}}) \quad (\text{A12.1.5})$$

$$= v^{\mathbf{a}} (u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}}) \quad (\text{A12.1.6})$$

hence Eq. (Q12.1.1). Meanwhile, Eqs. (1.2.6), (2.1.15) and (2.1.16) give

$$\mathcal{L}_u \omega_{\mathbf{a}} = u^{\mathbf{b}} (\nabla_{\mathbf{b}} \omega_{\mathbf{a}} - \nabla_{\mathbf{a}} \omega_{\mathbf{b}}) + \nabla_{\mathbf{a}} (u^{\mathbf{b}} \omega_{\mathbf{b}}) \quad (\text{A12.1.7})$$

$$= u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \quad (\text{A12.1.8})$$

in agreement with Eq. (Q12.1.1).

- Q12.2. (a) Derive Eq. (2.3.21).
 (b) Show that a coordinate basis vector $e_{\alpha}^{\mathbf{a}}$ is a Killing vector if and only if

$$\nabla_{\alpha} g_{\beta\gamma} = 0 \quad (\text{Q12.2.1})$$

for all β, γ , and explain the difference between $\nabla_{\alpha} g_{\beta\gamma}$ and $\nabla_{\mathbf{a}} g_{\mathbf{b}\mathbf{c}}$.

- (c) Show that a particle with momentum

$$p_{\mathbf{a}} = m g_{\mathbf{a}\mathbf{b}} \frac{dx^{\mathbf{b}}}{dt} \quad (\text{Q12.2.2})$$

and moving freely in a space with Killing vector field $\xi^{\mathbf{a}}$ has conserved quantity $\xi^{\mathbf{a}} p_{\mathbf{a}}$.

A12.2. (a) Using the Leibniz rule and Eq. (2.3.20),

$$u^a v^b \mathcal{L}_\xi g_{ab} = \mathcal{L}_\xi (u^a v^b g_{ab}) - g_{ab} v^b \mathcal{L}_\xi u^a - g_{ab} u^a \mathcal{L}_\xi v^b \quad (\text{A12.2.1})$$

$$= \xi^c \nabla_c (u^a v^b g_{ab}) - g_{ab} v^b (\xi^c \nabla_c u^a - u^c \nabla_c \xi^a) \\ - g_{ab} u^a (\xi^c \nabla_c v^b - v^c \nabla_c \xi^b) \quad (\text{A12.2.2})$$

$$= g_{ab} v^b u^c \nabla_c \xi^a + g_{ab} u^a v^c \nabla_c \xi^b \quad (\text{A12.2.3})$$

$$= u^a v^b (\nabla_a \xi_b + \nabla_b \xi_a) \quad (\text{A12.2.4})$$

(b) Using Eqs. (2.3.21), (2.2.7) and (2.2.10),

$$\mathcal{L}_{e_a} g_{ab} = \nabla_a (g_{bc} e_\alpha^c) + \nabla_b (g_{ac} e_\alpha^c) \quad (\text{A12.2.5})$$

$$= g_{bc} \Gamma_{a\alpha}^c + g_{ac} \Gamma_{b\alpha}^c \quad (\text{A12.2.6})$$

$$= \frac{1}{2} e_b^\beta e_a^\gamma (g_{\beta\gamma,\alpha} + g_{\beta\alpha,\gamma} - g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} - g_{\beta\alpha,\gamma}) \quad (\text{A12.2.7})$$

$$= e_b^\beta e_a^\gamma \nabla_\alpha g_{\beta\gamma} \quad (\text{A12.2.8})$$

$\nabla_\alpha g_{\beta\gamma}$ is the partial derivative of the metric components with respect to the coordinate x^α and is coordinate dependent, while $\nabla_a g_{bc}$ is the covariant derivative of the metric tensor and

$$\nabla_a g_{bc} \equiv 0 \quad (\text{A12.2.9})$$

(c) Using Eqs. (Q12.2.2) and (A12.2.5),

$$\frac{d}{dt} (\xi^a p_a) = m \frac{d}{dt} \left(\xi^a g_{ab} \frac{dx^b}{dt} \right) \quad (\text{A12.2.10})$$

$$= m \frac{dx^b}{dt} \frac{dx^c}{dt} \nabla_c (\xi^a g_{ab}) + m \xi^a g_{ab} \frac{d^2 x^b}{dt^2} \quad (\text{A12.2.11})$$

$$= \frac{1}{2} m \frac{dx^a}{dt} \frac{dx^b}{dt} \mathcal{L}_\xi g_{ab} + m \xi^a g_{ab} \frac{d^2 x^b}{dt^2} \quad (\text{A12.2.12})$$