Homework 13 - Symmetry

Q13.1. (a) Express $\mathcal{L}_u v^{\mathbf{a}}$ in a coordinate basis and deduce that it is independent of the metric.

(b) Use Eq. (2.3.20) to show that

$$\mathcal{L}_u \omega_{\mathbf{a}} = u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \tag{Q13.1.1}$$

and check that this is consistent with Eq. (1.2.6).

A13.1. (a) Using Eqs. (2.3.20), (2.2.8) and (Q11.1.2), in a coordinate basis,

$$\begin{aligned} \mathcal{L}_u v^{\mathbf{a}} &= u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} && \text{(A13.1.1)} \\ &= u^{\beta} \left(\frac{\partial v^{\alpha}}{\partial x^{\beta}} e_{\alpha}^{\mathbf{a}} + v^{\alpha} \Gamma^{\gamma}_{\beta\alpha} e_{\gamma}^{\mathbf{a}} \right) - v^{\beta} \left(\frac{\partial u^{\alpha}}{\partial x^{\beta}} e_{\alpha}^{\mathbf{a}} + u^{\alpha} \Gamma^{\gamma}_{\beta\alpha} e_{\gamma}^{\mathbf{a}} \right) && \text{(A13.1.2)} \\ &= \left(u^{\beta} \frac{\partial v^{\alpha}}{\partial x^{\beta}} - v^{\beta} \frac{\partial u^{\alpha}}{\partial x^{\beta}} \right) e_{\alpha}^{\mathbf{a}} && \text{(A13.1.3)} \end{aligned}$$

which is independent of the metric.

(b) Using the Leibniz rule and Eq. (2.3.20),

$$\begin{aligned} v^{\mathbf{a}} \mathcal{L}_u \omega_{\mathbf{a}} &= \mathcal{L}_u (\omega_{\mathbf{a}} v^{\mathbf{a}}) - \omega_{\mathbf{a}} \mathcal{L}_u v^{\mathbf{a}} && \text{(A13.1.4)} \\ &= u^{\mathbf{b}} \nabla_{\mathbf{b}} (\omega_{\mathbf{a}} v^{\mathbf{a}}) - \omega_{\mathbf{a}} \left(u^{\mathbf{b}} \nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}} \nabla_{\mathbf{b}} u^{\mathbf{a}} \right) && \text{(A13.1.5)} \\ &= v^{\mathbf{a}} \left(u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} \right) && \text{(A13.1.6)} \end{aligned}$$

hence Eq. (Q13.1.1). Meanwhile, Eqs. (1.2.6), (2.1.15) and (2.1.16) give

$$\begin{aligned} \mathcal{L}_u \omega_{\mathbf{a}} &= u^{\mathbf{b}} (\nabla_{\mathbf{b}} \omega_{\mathbf{a}} - \nabla_{\mathbf{a}} \omega_{\mathbf{b}}) + \nabla_{\mathbf{a}} \left(u^{\mathbf{b}} \omega_{\mathbf{b}} \right) && \text{(A13.1.7)} \\ &= u^{\mathbf{b}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + \omega_{\mathbf{b}} \nabla_{\mathbf{a}} u^{\mathbf{b}} && \text{(A13.1.8)} \end{aligned}$$

in agreement with Eq. (Q13.1.1).

Q13.2. (a) Derive Eq. (2.3.21).

(b) Show that a coordinate basis vector $e_{\alpha}^{\mathbf{a}}$ is a Killing vector if and only if

$$\nabla_{\alpha} g_{\beta\gamma} = 0 \tag{Q13.2.1}$$

for all β, γ, and explain the difference between $\nabla_{\alpha} g_{\beta\gamma}$ and $\nabla_{\mathbf{a}} g_{\mathbf{bc}}$.

(c) Show that a particle with momentum

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \tag{Q13.2.2}$$

and moving freely in a space with Killing vector field $\xi^{\mathbf{a}}$ has conserved quantity $\xi^{\mathbf{a}} p_{\mathbf{a}}$.

A13.2. (a) Using the Leibniz rule and Eq. (2.3.20),

$$\begin{aligned} u^{\mathbf{a}} v^{\mathbf{b}} \mathcal{L}_{\xi} g_{\mathbf{ab}} &= \mathcal{L}_{\xi} \left(u^{\mathbf{a}} v^{\mathbf{b}} g_{\mathbf{ab}} \right) - g_{\mathbf{ab}} v^{\mathbf{b}} \mathcal{L}_{\xi} u^{\mathbf{a}} - g_{\mathbf{ab}} u^{\mathbf{a}} \mathcal{L}_{\xi} v^{\mathbf{b}} && \text{(A13.2.1)} \\ &= \xi^{\mathbf{c}} \nabla_{\mathbf{c}} \left(u^{\mathbf{a}} v^{\mathbf{b}} g_{\mathbf{ab}} \right) - g_{\mathbf{ab}} v^{\mathbf{b}} \left(\xi^{\mathbf{c}} \nabla_{\mathbf{c}} u^{\mathbf{a}} - u^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{a}} \right) \\ &\quad - g_{\mathbf{ab}} u^{\mathbf{a}} \left(\xi^{\mathbf{c}} \nabla_{\mathbf{c}} v^{\mathbf{b}} - v^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{b}} \right) && \text{(A13.2.2)} \\ &= g_{\mathbf{ab}} v^{\mathbf{b}} u^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{a}} + g_{\mathbf{ab}} u^{\mathbf{a}} v^{\mathbf{c}} \nabla_{\mathbf{c}} \xi^{\mathbf{b}} && \text{(A13.2.3)} \\ &= u^{\mathbf{a}} v^{\mathbf{b}} \left(\nabla_{\mathbf{a}} \xi_{\mathbf{b}} + \nabla_{\mathbf{b}} \xi_{\mathbf{a}} \right) && \text{(A13.2.4)} \end{aligned}$$

(b) Using Eqs. (2.3.21), (2.2.7) and (2.2.10),

$$\begin{aligned} \mathcal{L}_{e_\alpha} g_{\mathbf{ab}} &= \nabla_{\mathbf{a}} \left(g_{\mathbf{bc}} e^{\mathbf{c}}_{\alpha} \right) + \nabla_{\mathbf{b}} \left(g_{\mathbf{ac}} e^{\mathbf{c}}_{\alpha} \right) && \text{(A13.2.5)} \\ &= g_{\mathbf{bc}} \Gamma^{\mathbf{c}}_{\mathbf{a}\alpha} + g_{\mathbf{ac}} \Gamma^{\mathbf{c}}_{\mathbf{b}\alpha} && \text{(A13.2.6)} \\ &= \frac{1}{2} e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{a}} \left(g_{\beta\gamma,\alpha} + g_{\beta\alpha,\gamma} - g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} - g_{\beta\alpha,\gamma} \right) && \text{(A13.2.7)} \\ &= e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{a}} \nabla_{\alpha} g_{\beta\gamma} && \text{(A13.2.8)} \end{aligned}$$

$\nabla_\alpha g_{\beta\gamma}$ is the partial derivative of the metric components with respect to the coordinate x^α and is coordinate dependent, while $\nabla_{\mathbf{a}} g_{\mathbf{bc}}$ is the covariant derivative of the metric tensor and

$$\nabla_{\mathbf{a}} g_{\mathbf{bc}} \equiv 0 \qquad \text{(A13.2.9)}$$

(c) Using Eqs. (Q13.2.2) and (A13.2.5),

$$\begin{aligned} \frac{d}{dt} \left(\xi^{\mathbf{a}} p_{\mathbf{a}} \right) &= m \frac{d}{dt} \left(\xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \right) && \text{(A13.2.10)} \\ &= m \frac{dx^{\mathbf{b}}}{dt} \frac{dx^{\mathbf{c}}}{dt} \nabla_{\mathbf{c}} \left(\xi^{\mathbf{a}} g_{\mathbf{ab}} \right) + m \xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{d^2 x^{\mathbf{b}}}{dt^2} && \text{(A13.2.11)} \\ &= \frac{1}{2} m \frac{dx^{\mathbf{a}}}{dt} \frac{dx^{\mathbf{b}}}{dt} \mathcal{L}_{\xi} g_{\mathbf{ab}} + m \xi^{\mathbf{a}} g_{\mathbf{ab}} \frac{d^2 x^{\mathbf{b}}}{dt^2} && \text{(A13.2.12)} \end{aligned}$$