Chapter 1

Differential Topology

1.1 Tensors

A **tensor** is a mathematical object that directly represents a physical quantity. Tensors of the same type can be added, and multiplied by a scalar, in the usual way. Scalars and vectors are tensors, but many physical quantities are some other type of tensor.

A **scalar** is a tensor that behaves like a number. Examples of spatial[1] scalars are time t, energy E and electric potential ϕ. Examples of spacetime scalars are proper time τ, mass m and charge q.

1.1.1 Vectors and covectors

A **vector** is a tensor that behaves like an arrow. Their properties inspire the vector

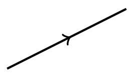

Figure 1.1.1: A vector.

space axioms of mathematics. A scalar times a vector is a vector and the sum of two vectors is a vector, see Figure 1.1.2. Examples of vectors are displacement $\vec{dx}$, velocity

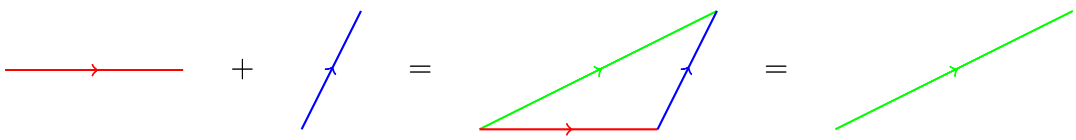

Figure 1.1.2: The sum of two vectors is a vector.

[1]Physical quantities may be one type of tensor with respect to one space but another type of tensor with respect to another space. For example, a displacement in time is a scalar with respect to space but a vector with respect to time. Unless otherwise specified, the space can be assumed to be space, or spacetime in the context of relativity.

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \tag{1.1.1}$$

and acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \tag{1.1.2}$$

A **covector** or **one-form** is a tensor that behaves like the local linearized form of contour lines or the crests of a wave, see Figure 1.1.3. A scalar times a covector

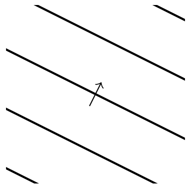

Figure 1.1.3: A covector.

is a covector and the sum of two covectors is a covector, see Figure 1.1.4. Examples

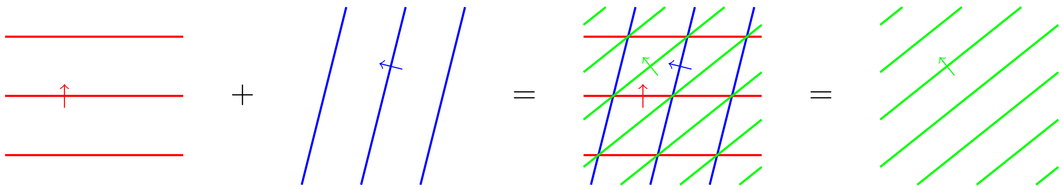

Figure 1.1.4: The sum of two covectors is a covector.

of covectors are the gradient of a scalar field $\underline{\nabla}\phi$, wave "vectors" $\underline{k}$, electric field and magnetic "vector" potential

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \tag{1.1.3}$$

momentum

$$\underline{p} = \hbar \underline{k} \tag{1.1.4}$$

and force

$$\underline{F} = q\underline{E} \tag{1.1.5}$$

or

$$\underline{F} = \frac{d\underline{p}}{dt} \tag{1.1.6}$$

A vector can be **contracted** with a covector to give a scalar

$$\vec{v} \cdot \underline{\omega} = \text{scalar} \tag{1.1.7}$$

corresponding to the number of covector planes crossed by the vector, with sign given by the relative orientations of the vector and covector, see Figure 1.1.5. For example, a

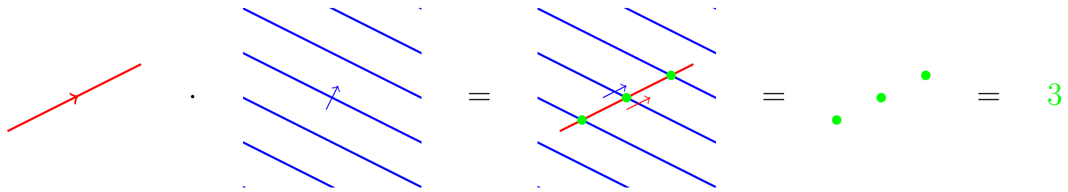

Figure 1.1.5: A vector contracted with a covector gives a scalar.

displacement contracted with the gradient of a scalar field gives the change in the scalar field

$$\vec{dx} \cdot \underline{\nabla}\phi = d\phi \tag{1.1.8}$$

and power equals force contracted with velocity

$$P = \underline{F} \cdot \vec{v} \tag{1.1.9}$$

Comparing vectors and covectors, the magnitude of a vector is given by its length, while the magnitude of a covector is given by the density of its planes. The direction of a vector is along its length (intrinsically oriented), while the direction of a covector is normal to its planes (extrinsically oriented) in the sense that $\underline{n} \cdot \vec{v} = 0$ for any vector $\vec{v}$ lying in the plane of the covector $\underline{n}$. Thus, a vector is an intrinsically oriented dimension one plane element, while a covector is an extrinsically oriented codimension[2] one plane density.

1.1.2 Exterior algebra

Vectors or covectors can be multiplied together using the **exterior** or **wedge product**, generating **multivectors** or **differential forms** respectively. Multivectors, differential forms and their exterior algebra have an elegant mathematical structure and clear physical interpretation.

The exterior product of two vectors is the **two-vector** given by the oriented plane element formed by the two vectors, see Figure 1.1.6. Note that the shape of the plane

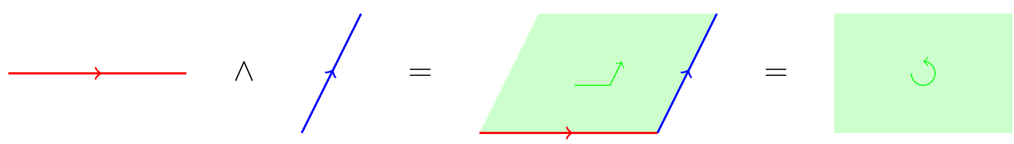

Figure 1.1.6: The exterior product of two vectors is a two-vector.

element does not matter

$$(2\vec{a}) \wedge \vec{b} = \vec{a} \wedge (2\vec{b}) = 2(\vec{a} \wedge \vec{b}) \tag{1.1.10}$$

[2]Codimension d is dimension $D - d$ where D is the dimension of the space.

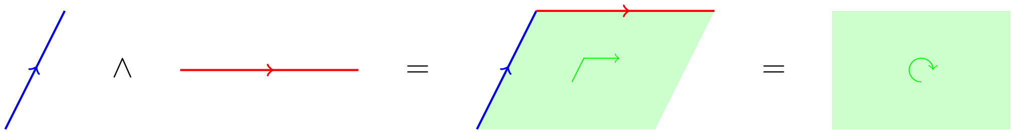

Figure 1.1.7: The exterior product is antisymmetric.

only its plane, area and orientation. The exterior product is antisymmetric

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} \tag{1.1.11}$$

since swapping the vectors in Figure 1.1.6 would reverse the orientation, see Figure 1.1.7. A scalar times a two-vector is a two-vector and the sum of two two-vectors is a two-vector,

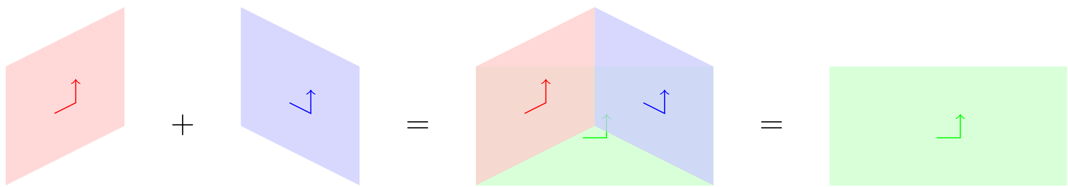

Figure 1.1.8: The sum of two two-vectors is a two-vector.

see Figure 1.1.8. Examples of two-vectors are surface element $d\vec{\vec{S}}$, angular momentum[3]

$$\vec{\vec{L}} = m\,\vec{x} \wedge \vec{v} \tag{1.1.12}$$

and torque

$$\vec{\vec{\tau}} = \frac{d\vec{\vec{L}}}{dt} \tag{1.1.13}$$

The exterior product of two one-forms is a **two-form** given by the oriented intersections of the one-form planes, see Figure 1.1.9. Note that the position of the intersections

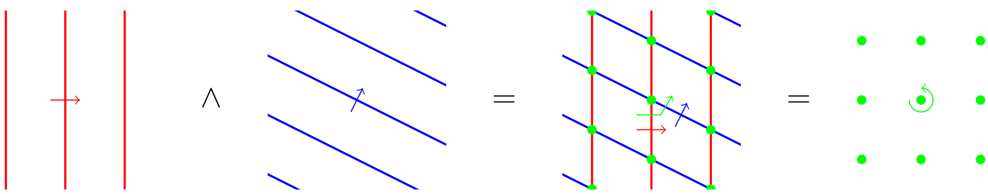

Figure 1.1.9: The exterior product of two one-forms is a two-form.

does not matter, only their density and orientation. A scalar times a two-form is a two-form and the sum of two two-forms is a two-form too, see Figure 1.1.10. Examples of

[3]Note that $\vec{x}$ is a vector, and hence $\vec{\vec{L}}$ is a two-vector, only in flat space. We will onsider more general spaces in Section 1.2.

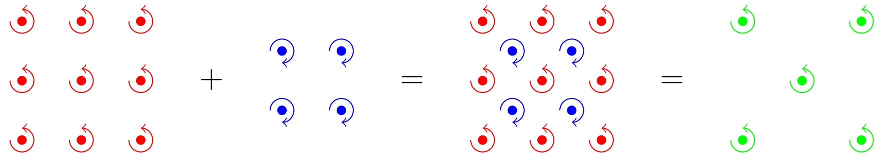

Figure 1.1.10: The sum of two two-forms is a two-form.

two-forms are magnetic flux density

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \tag{1.1.14}$$

and electric current density

$$\underline{\underline{j}} = \underline{\underline{\rho}} \cdot \vec{v} \tag{1.1.15}$$

A two-form can be contracted with a two-vector to give a scalar,

$$\underline{\underline{\omega}} \cdot \vec{\vec{v}} = \text{scalar} \tag{1.1.16}$$

see Figure 1.1.11. For example, an electric current density contracted with an area

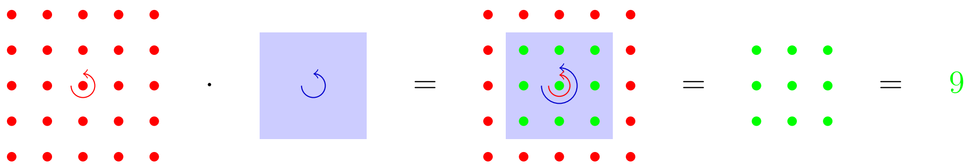

Figure 1.1.11: A two-form contracted with a two-vector gives a scalar.

element gives the current

$$\underline{\underline{j}} \cdot d\vec{\vec{S}} = dI \tag{1.1.17}$$

In general, an n-vector is an intrinsically oriented dimension n plane element, and an n-form is an extrinsically oriented codimension n plane density. The wedge product of an m-form with an n-form is an $(m+n)$-form, and similarly for multivectors. An m-form contracted with an n-vector is an $(m-n)$-form or an $(n-m)$-vector.

The dimension of the space of n-forms is $N!/[n!(N-n)!]$, where N is the dimension of the space, and $0 \leq n \leq N$ due to the antisymmetry of the exterior product. For example, in three dimensions, there are $1, 3, 3, 1$ independent $0, 1, 2, 3$-forms, respectively, and no higher forms. The same is true for multivectors. The multivectors and differential forms in three dimensions are shown in Figure 1.1.12.

The antisymmetry of the exterior product gives, for an m-form $\boldsymbol{\omega}$ and an l-form $\boldsymbol{\sigma}$,

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = (-1)^{ml} \boldsymbol{\sigma} \wedge \boldsymbol{\omega} \tag{1.1.18}$$

and similarly for multivectors. Also, for an m-form $\boldsymbol{\omega}$ and an n-vector $\boldsymbol{v}$,

$$\boldsymbol{\omega} \cdot \boldsymbol{v} = \begin{cases} (-1)^{(m-n)n}\, \boldsymbol{v} \cdot \boldsymbol{\omega} & \text{for } n \leq m \\ (-1)^{(n-m)m}\, \boldsymbol{v} \cdot \boldsymbol{\omega} & \text{for } n \geq m \end{cases} \tag{1.1.19}$$

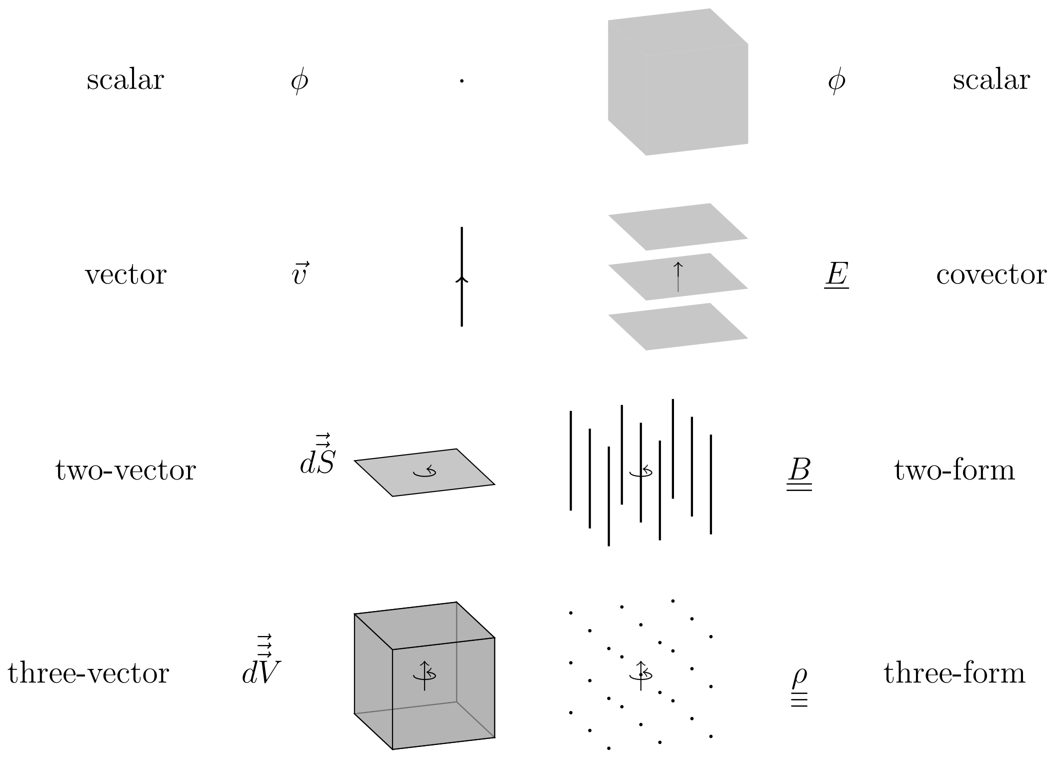

Figure 1.1.12: Multivectors and differential forms in three dimensions.

Exterior product is associative

$$\boldsymbol{\omega} \wedge (\boldsymbol{\sigma} \wedge \boldsymbol{\rho}) = (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \wedge \boldsymbol{\rho} \tag{1.1.20}$$

but contraction is generally not associative

$$\boldsymbol{u} \cdot (\boldsymbol{\omega} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{v} \quad \text{for } \deg \boldsymbol{u} + \deg \boldsymbol{v} \leq \deg \boldsymbol{\omega} \tag{1.1.21}$$

For n-vector $\boldsymbol{v}$, m-form $\boldsymbol{\omega}$ and l-form $\boldsymbol{\sigma}$

$$\boldsymbol{v} \cdot (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = \begin{cases} (\boldsymbol{v} \cdot \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{ml} (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \wedge \boldsymbol{\omega} & \text{for } 1 = n \leq m, l \\ (\boldsymbol{\omega} \cdot \boldsymbol{v}) \cdot \boldsymbol{\sigma} + (-1)^{l} (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \wedge \boldsymbol{\omega} & \text{for } 1 = m \leq n \leq l \\ (\boldsymbol{v} \cdot \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{m} (\boldsymbol{\sigma} \cdot \boldsymbol{v}) \cdot \boldsymbol{\omega} & \text{for } 1 = l \leq n \leq m \\ (\boldsymbol{\omega} \cdot \boldsymbol{v}) \cdot \boldsymbol{\sigma} + (-1)^{ml} (\boldsymbol{\sigma} \cdot \boldsymbol{v}) \cdot \boldsymbol{\omega} & \text{for } l + m - 1 = n \\ (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\omega} & \text{for } l + m \leq n \end{cases} \tag{1.1.22}$$

If $1 < l, m, n < l + m - 1$ then one gets partial contractions of $\boldsymbol{v}$ with both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$, for example

$$\vec{\vec{v}} \cdot \left(\underline{\underline{\omega}} \wedge \underline{\sigma}\right) = \left(\underline{\underline{\omega}} \cdot \vec{\vec{v}}\right) \underline{\sigma} + \left(\underline{\underline{\omega}} \cdot \vec{\vec{v}} \cdot \underline{\sigma} + \underline{\sigma} \cdot \vec{\vec{v}} \cdot \underline{\underline{\omega}}\right) + \left(\underline{\sigma} \cdot \vec{\vec{v}}\right) \underline{\underline{\omega}} \tag{1.1.23}$$

where the middle two terms should be combined to form a two-form.