

1.2 Topological tensor calculus

1.2.1 Tensor fields

Finite displacements in Euclidean space can be represented by arrows and have a natural vector space structure, but finite displacements in more general curved spaces, such as on the surface of a sphere, do not. However, an infinitesimal neighborhood of a point in a smooth curved space¹ looks like an infinitesimal neighborhood of Euclidean space, and infinitesimal displacements \vec{dx} retain the vector space structure of displacements in Euclidean space. An infinitesimal neighborhood of a point can be infinitely rescaled to generate a finite vector space, called the **tangent space**, at the point. A vector lives in the tangent space of a point. Note that vectors do not stretch from one point to

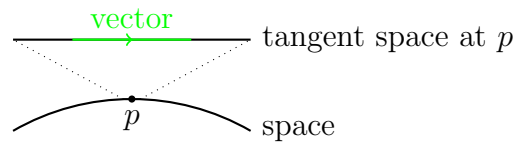


Figure 1.2.1: A vector in the tangent space of a point.

another, and vectors at different points live in different tangent spaces and so cannot be added.

For example, rescaling the infinitesimal displacement \vec{dx} by dividing it by the infinitesimal scalar dt gives the velocity

$$\vec{v} = \frac{\vec{dx}}{dt} \quad (1.2.1)$$

which is a vector. Similarly, we can picture the covector $\nabla\phi$ as the infinitesimal contours of ϕ in a neighborhood of a point, infinitely rescaled to generate a finite covector in the point's cotangent space. More generally, infinitely rescaling the neighborhood of a point generates the **tensor space** and its algebra at the point. The tensor space contains the tangent and cotangent spaces as vector subspaces.

A **tensor field** is something that takes tensor values at every point in a space. Tensor fields of the same type can be added, and multiplied by a scalar, in the usual way.

1.2.2 Exterior derivative

The **exterior derivative**²

$$\nabla \wedge \omega \quad (1.2.2)$$

of a differential form ω is defined as the topological (antisymmetric) derivative so that the exterior derivative of a differential form is also a differential form. It does not

¹In mathematical language, a smooth manifold.

²The mathematical notation for $\nabla \wedge \omega$ is $d\omega$.

depend on how the tensor spaces at different points are connected. This makes the exterior derivative $\nabla \wedge$ simpler than the more general covariant derivative ∇ defined later, and gives it a clear physical interpretation.

For example, a scalar field can be thought of as a codimension zero plane density, and its exterior derivative is the one-form field given by the oriented edges of the scalar field's codimension zero planes, i.e. the contours of the scalar field. The exterior derivative of a one-form field is the two-form field given by the oriented edges of the one-form field's planes. See Figure 1.2.2. More generally, the exterior derivative of an n -form field is

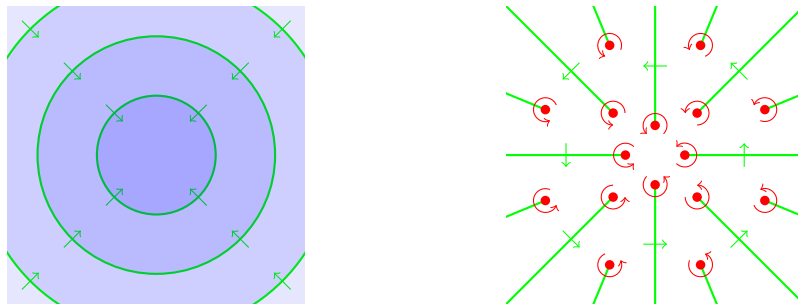


Figure 1.2.2: Left: $\nabla \wedge \phi = \underline{\xi}$, right: $\nabla \wedge \underline{\zeta} = \underline{\underline{\rho}}$.

the $(n + 1)$ -form field given by the oriented edges of the n -form field's codimension n planes. Thus the exterior derivative $\nabla \wedge$ has the meaning ‘the oriented boundaries of’ and gives a measure of the spacial rate of change of the tensor field.

The exterior derivative has the defining properties:

- Acting on a scalar field, the exterior derivative is equal to the gradient

$$\nabla \wedge \omega = \nabla \omega \tag{1.2.3}$$

- For any differential form field ω ,

$$\nabla \wedge \nabla \wedge \omega = 0 \tag{1.2.4}$$

since the boundary of a boundary is zero, as can be seen from Figure 1.2.2.

- Taking into account the antisymmetry of the wedge product, the Leibnitz rule is

$$\nabla \wedge (\omega \wedge \sigma) = (\nabla \wedge \omega) \wedge \sigma + (-1)^{\text{deg } \omega} \omega \wedge (\nabla \wedge \sigma) \tag{1.2.5}$$

The exterior derivative can be used to define the **Lie derivative** of a differential form field ω with respect to a vector field v

$$\mathcal{L}_v \omega = v \cdot (\nabla \wedge \omega) + \nabla \wedge (v \cdot \omega) \tag{1.2.6}$$

The Lie derivative will be discussed in Section 2.3.3.

1.2.3 Integration

An n -form field ω naturally contracts with an n -dimensional surface S to give a scalar

$$\int_S \omega = \text{scalar} \quad (1.2.7)$$

with the same interpretation as the contraction of an n -form with an n -vector. If we divide the surface S into infinitesimal surface elements $d\mathbf{S}$, the integral of ω over S can be written in the more familiar form

$$\int_S \omega \cdot d\mathbf{S} \quad (1.2.8)$$

For example, the integral of a current density \underline{j} over a surface S is the current I flowing through the surface

$$I = \int_S \underline{j} \cdot d\vec{S} \quad (1.2.9)$$

or integral of a charge density $\underline{\rho}$ over a volume V is the charge Q contained in the volume

$$Q = \int_V \underline{\rho} \cdot d\vec{V} \quad (1.2.10)$$

Stokes' theorem states that

$$\int_S \nabla \wedge \omega = \int_{\partial S} \omega \quad (1.2.11)$$

where ∂S is the boundary of S , see Figure 1.2.3.

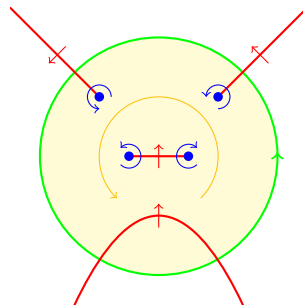


Figure 1.2.3: Stokes' theorem: $\int_S \nabla \wedge \omega = \int_{\partial S} \omega = 2$.