

1.3 Electromagnetism

1.3.1 Maxwell's equations

Maxwell's equations are naturally expressed in terms of differential forms

$$\begin{aligned}\underline{\nabla} \wedge \underline{B} &= 0 & , & \quad \underline{\nabla} \wedge \underline{E} + \underline{\dot{B}} = 0 \\ \underline{\nabla} \wedge \underline{D} &= \underline{\rho} & , & \quad \underline{\nabla} \wedge \underline{H} - \underline{\dot{D}} = \underline{j}\end{aligned}\tag{1.3.1}$$

The electric displacement field (electric flux density) \underline{D} is related to the electric field \underline{E} by

$$\underline{D} = \underline{\vec{\epsilon}}_0 \cdot \underline{E} + \underline{P}\tag{1.3.2}$$

where \underline{P} is the polarization (electric dipole) density, and the magnetic field \underline{H} is related to the magnetic flux density \underline{B} by

$$\underline{H} = \underline{\vec{\mu}}_0^{-1} \cdot \underline{B} - \underline{M}\tag{1.3.3}$$

where \underline{M} is the magnetization (magnetic dipole) density.¹

The electromagnetic energy density is

$$\underline{u} = \frac{1}{2} \underline{E} \wedge \underline{D} + \frac{1}{2} \underline{B} \wedge \underline{H}\tag{1.3.4}$$

and the Poynting vector (electromagnetic energy flux density) is

$$\underline{S} = \underline{E} \wedge \underline{H}\tag{1.3.5}$$

The first line of Eq. (1.3.1) is automatic given

$$\underline{E} = -\underline{\nabla} \wedge \phi - \underline{\dot{A}}\tag{1.3.6}$$

$$\underline{B} = \underline{\nabla} \wedge \underline{A}\tag{1.3.7}$$

which has gauge invariance

$$\phi \rightarrow \phi + \dot{\lambda}\tag{1.3.8}$$

$$\underline{A} \rightarrow \underline{A} - \underline{\nabla} \wedge \lambda\tag{1.3.9}$$

while the second line of Eq. (1.3.1) gives

$$\underline{\dot{\rho}} + \underline{\nabla} \wedge \underline{j} = 0\tag{1.3.10}$$

¹The tensors $\underline{\vec{\epsilon}}_0$ and $\underline{\vec{\mu}}_0^{-1}$ will be discussed later.

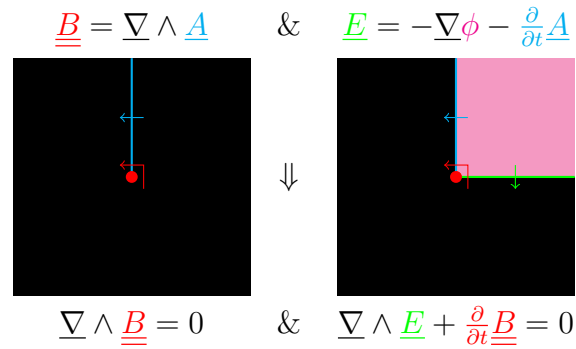


Figure 1.3.1: Electric field and magnetic flux induced by electric and magnetic potentials. One internal space dimension has been suppressed.

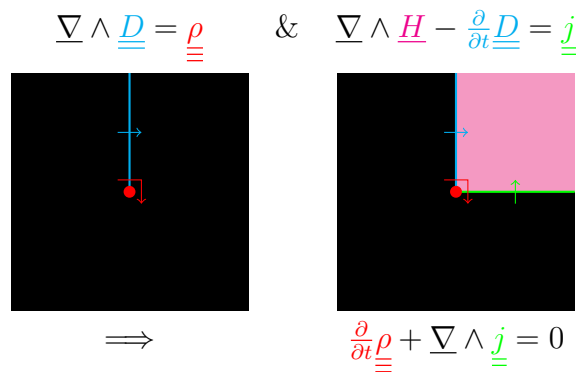


Figure 1.3.2: Electric flux and magnetic field induced by charge and current. One external space dimension has been suppressed.

1.3.2 Spacetime decomposition

The 3+1 dimensional description of Maxwell's equations above requires a choice of time coordinate t in 4-dimensional spacetime. The one-form field

$$\underline{e}^t = \underline{\nabla} \wedge t \quad (1.3.11)$$

then represents the corresponding choice of 3-dimensional spatial hypersurfaces in 4-dimensional spacetime. One can also introduce a vector field \vec{e}_t satisfying

$$\vec{e}_t \cdot \underline{e}^t = 1 \quad (1.3.12)$$

corresponding to the 1-dimensional time lines that define the spatial rest-frame. A spacetime displacement then decomposes as

$$\vec{dx} = dt \vec{e}_t + \vec{dx}_3 \quad (1.3.13)$$

where the spatial displacement satisfies

$$\underline{e}^t \cdot \vec{dx}_3 = 0 \quad (1.3.14)$$

i.e. spatial displacements lie within the spatial hypersurfaces.

More generally, an n -form ω can be decomposed into an $(n-1)$ -form $\omega^{(1)}$ and an n -form $\omega^{(3)}$ as

$$\omega = e^t \wedge \omega^{(1)} + \omega^{(3)} \quad (1.3.15)$$

with

$$e_t \cdot \omega^{(1)} = 0 \quad (1.3.16)$$

$$e_t \cdot \omega^{(3)} = 0 \quad (1.3.17)$$

and similarly for n -vectors. Inverting gives

$$\omega^{(1)} = e_t \cdot \omega \quad (1.3.18)$$

$$\omega^{(3)} = e_t \cdot (e^t \wedge \omega) \quad (1.3.19)$$

and so we can also express this decomposition as

$$\omega = e^t \wedge (e_t \cdot \omega) + e_t \cdot (e^t \wedge \omega) \quad (1.3.20)$$

For example, the spacetime current density decomposes as

$$\underline{\underline{J}} = \underline{e}^t \wedge \underline{\underline{j}} - \underline{\underline{\rho}} \quad (1.3.21)$$

where the spatial current density and charge density

$$\underline{\underline{j}} = \vec{e}_t \cdot \underline{\underline{J}} \quad (1.3.22)$$

$$\underline{\underline{\rho}} = -\vec{e}_t \cdot (\underline{e}^t \wedge \underline{\underline{J}}) \quad (1.3.23)$$

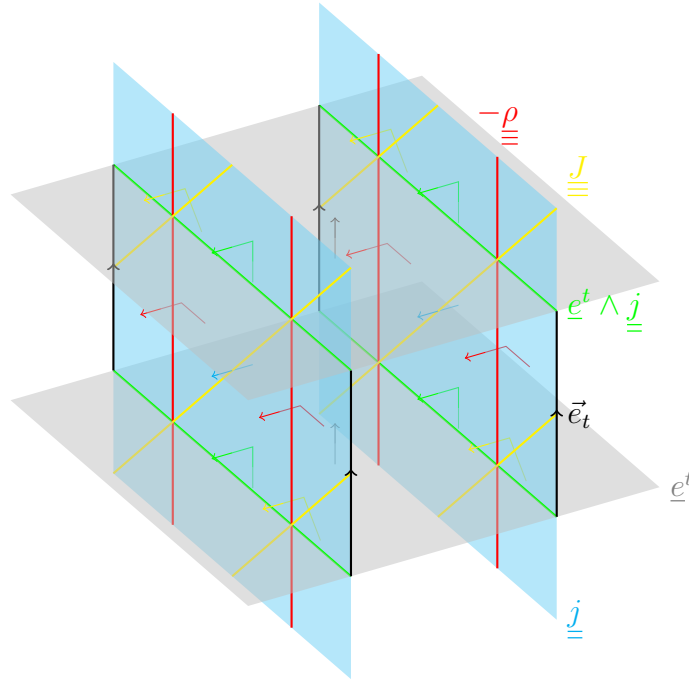


Figure 1.3.3: An illustration of the 3+1 decomposition of $\underline{\underline{J}}$ given in Eq. (1.3.21). One spatial dimension, and densities and orientations in that dimension, has been suppressed. See J.pdf for a rotatable version of this figure.

satisfy

$$\vec{e}_t \cdot \underline{\underline{j}} = 0 \tag{1.3.24}$$

$$\vec{e}_t \cdot \underline{\underline{\rho}} = 0 \tag{1.3.25}$$

see Figure 1.3.3.

Defining the time and spatial exterior derivatives

$$\dot{\omega} \equiv \mathcal{L}_{e_t} \omega = e_t \cdot (\nabla \wedge \omega) + \nabla \wedge (e_t \cdot \omega) \tag{1.3.26}$$

$$\nabla^{(3)} \wedge \omega \equiv e_t \cdot (e^t \wedge \nabla \wedge \omega) - e^t \wedge \nabla \wedge (e_t \cdot \omega) \tag{1.3.27}$$

and using Eq. (1.3.20), the spacetime exterior derivative decomposes as

$$\nabla \wedge \omega = e^t \wedge \dot{\omega} + \nabla^{(3)} \wedge \omega \tag{1.3.28}$$

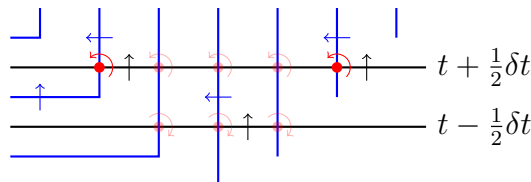


Figure 1.3.4: The topological interpretation of $e^t \wedge \dot{\omega}$ is given by the intersection of ω 's surfaces with the surface $t + \frac{1}{2}\delta t$ minus the intersection with the surface $t - \frac{1}{2}\delta t$, divided by δt .

Now

$$\nabla \wedge \underline{e}^t = \nabla \wedge \nabla \wedge t = 0 \quad (1.3.29)$$

therefore, using Eqs. (1.3.15), (1.2.5) and (1.3.28),

$$\nabla \wedge \omega = \nabla \wedge (\underline{e}^t \wedge \omega^{(1)} + \omega^{(3)}) \quad (1.3.30)$$

$$= (\nabla \wedge \underline{e}^t) \wedge \omega^{(1)} - \underline{e}^t \wedge \nabla \wedge \omega^{(1)} + \nabla \wedge \omega^{(3)} \quad (1.3.31)$$

$$= -\underline{e}^t \wedge \nabla^{(3)} \wedge \omega^{(1)} + \underline{e}^t \wedge \dot{\omega}^{(3)} + \nabla^{(3)} \wedge \omega^{(3)} \quad (1.3.32)$$

1.3.3 Relativistic electrodynamics

Maxwell's equations

To relativize Eqs. (1.3.6) and (1.3.7), we introduce the electromagnetic potential

$$\underline{A} = \phi \underline{e}^t - \underline{A}^{(3)} \quad (1.3.33)$$

and electromagnetic field strength

$$\underline{F} = \underline{e}^t \wedge \underline{E} - \underline{B} \quad (1.3.34)$$

then, using Eq. (1.3.32),

$$\underline{F} = \nabla \wedge \underline{A} \quad (1.3.35)$$

decomposes to

$$\underline{e}^t \wedge \underline{E} - \underline{B} = \underline{e}^t \wedge \left(-\nabla^{(3)} \phi - \dot{\underline{A}}^{(3)} \right) - \nabla^{(3)} \wedge \underline{A}^{(3)} \quad (1.3.36)$$

relativizing Eqs. (1.3.6) and (1.3.7). Eq. (1.3.35) has gauge invariance

$$\underline{A} \rightarrow \underline{A} + \nabla \wedge \lambda \quad (1.3.37)$$

which decomposes to

$$\phi \underline{e}^t - \underline{A}^{(3)} \rightarrow (\phi + \dot{\lambda}) \underline{e}^t - (\underline{A}^{(3)} - \nabla^{(3)} \lambda) \quad (1.3.38)$$

relativizing Eqs. (1.3.8) and (1.3.9), and implies

$$\nabla \wedge \underline{F} = 0 \quad (1.3.39)$$

which decomposes to

$$-\underline{e}^t \wedge \left(\nabla^{(3)} \wedge \underline{E} + \dot{\underline{B}} \right) - \nabla^{(3)} \wedge \underline{B} = 0 \quad (1.3.40)$$

relativizing the first line of Maxwell's equations, Eq. (1.3.1).

To get the second line of Maxwell's equations, we introduce the electromagnetic flux density

$$\underline{\underline{G}} = -\underline{e}^t \wedge \underline{H} - \underline{\underline{D}} \quad (1.3.41)$$

and spacetime current density, Eq. (1.3.21),

$$\underline{\underline{J}} = \underline{e}^t \wedge \underline{j} - \underline{\underline{\rho}} \quad (1.3.42)$$

Then, using Eq. (1.3.32),

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{J}} \quad (1.3.43)$$

decomposes to

$$\underline{e}^t \wedge \left(\underline{\nabla}^{(3)} \wedge \underline{H} - \underline{\dot{D}} \right) - \underline{\nabla}^{(3)} \wedge \underline{\underline{D}} = \underline{e}^t \wedge \underline{j} - \underline{\underline{\rho}} \quad (1.3.44)$$

relativizing the second line of Maxwell's equations, Eq. (1.3.1). Eq. (1.3.43) implies

$$\underline{\nabla} \wedge \underline{\underline{J}} = 0 \quad (1.3.45)$$

which decomposes to

$$-\underline{e}^t \wedge \left(\underline{\dot{\rho}} + \underline{\nabla}^{(3)} \wedge \underline{j} \right) = 0 \quad (1.3.46)$$

relativizing Eq. (1.3.10).

In summary, the relativistic form of Maxwell's equations is

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \quad (1.3.47)$$

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{J}} \quad (1.3.48)$$

with Eq. (1.3.47) implied by

$$\underline{\underline{F}} = \underline{\nabla} \wedge \underline{A} \quad (1.3.49)$$

which has gauge invariance

$$\underline{A} \rightarrow \underline{A} + \underline{\nabla} \wedge \lambda \quad (1.3.50)$$

and Eq. (1.3.48) implying

$$\underline{\nabla} \wedge \underline{\underline{J}} = 0 \quad (1.3.51)$$

These equations, and their implication of the equations in Section 1.3.1, are illustrated in Figures 1.3.5 and 1.3.6.

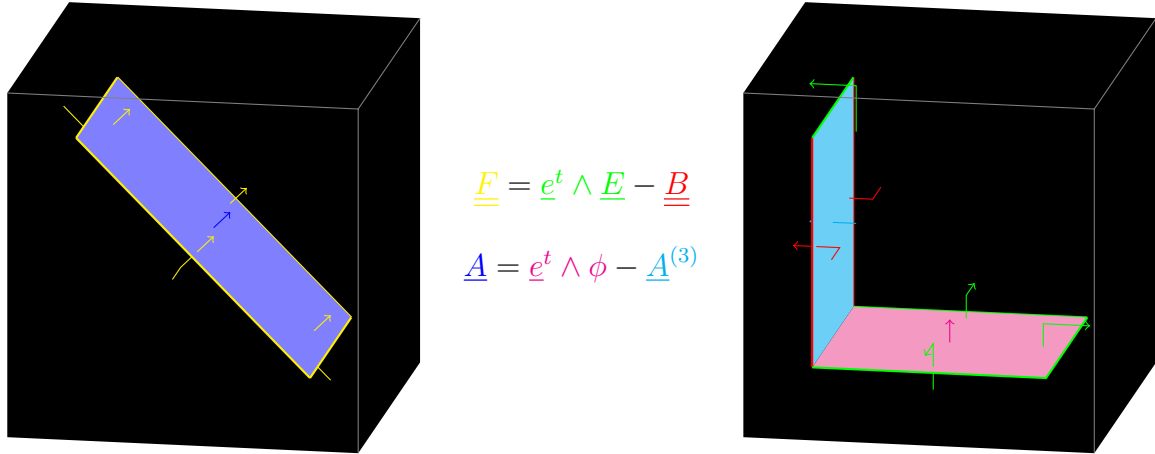


Figure 1.3.5: Space and time decomposition of $\underline{F} = \underline{\nabla} \wedge \underline{A}$ and $\underline{\nabla} \wedge \underline{F} = 0$ to give $\underline{B} = \underline{\nabla}^{(3)} \wedge \underline{A}^{(3)}$ and $\underline{e}^t \wedge \underline{E} = \underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \phi) - \underline{e}^t \wedge \frac{\partial}{\partial t} \underline{A}^{(3)}$ and $\underline{\nabla}^{(3)} \wedge \underline{B} = 0$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{E}) + \underline{e}^t \wedge \frac{\partial}{\partial t} \underline{B} = 0$. One internal space dimension has been suppressed. Compare Figure 1.3.1.

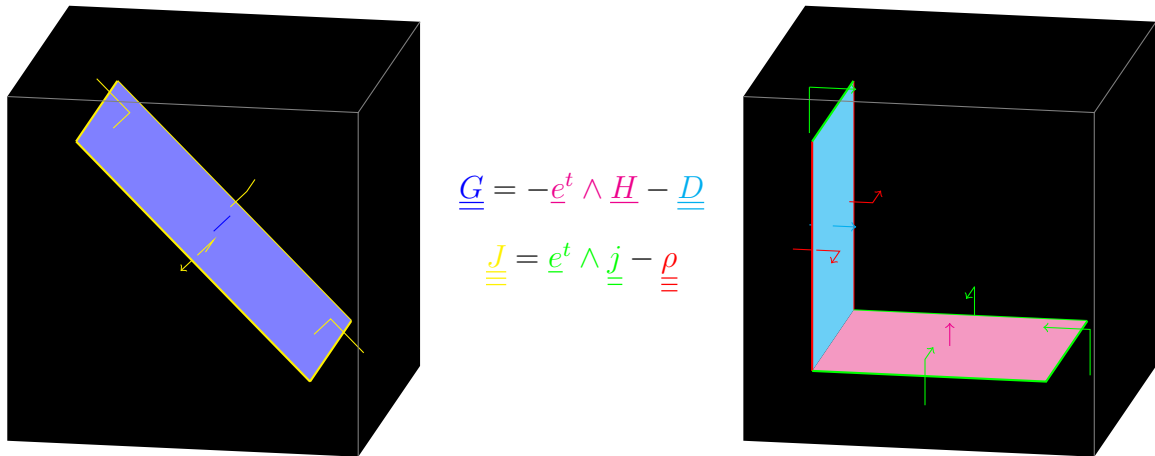


Figure 1.3.6: Space and time decomposition of $\underline{\nabla} \wedge \underline{G} = \underline{J}$ and $\underline{\nabla} \wedge \underline{J} = 0$ to give $\underline{\nabla}^{(3)} \wedge \underline{D} = \underline{\rho}$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{H}) - \underline{e}^t \wedge \frac{\partial}{\partial t} \underline{D} = \underline{e}^t \wedge \underline{j}$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{j}) + \underline{e}^t \wedge \frac{\partial}{\partial t} \underline{\rho} = 0$. One external space dimension has been suppressed. Compare Figure 1.3.2.

Lorentz force

Using the coordinate time t , we can define the spacetime velocity ²

$$\vec{v} = \frac{d\vec{x}}{dt} \quad (1.3.52)$$

which, following Eq. (1.3.13), decomposes as

$$\vec{v} = \vec{e}_t + \vec{v}_3 \quad (1.3.53)$$

and the spacetime force

$$\underline{f} = \frac{dp}{dt} \quad (1.3.54)$$

which decomposes as

$$\underline{f} = P\underline{e}^t - \underline{F} \quad (1.3.55)$$

Eqs. (1.3.34) and (1.3.53) give

$$\underline{F} \cdot \vec{v} = (\underline{e}^t \wedge \underline{E} - \underline{B}) \cdot (\vec{e}_t + \vec{v}_3) \quad (1.3.56)$$

$$= (\underline{E} \cdot \vec{v}_3) \underline{e}^t - (\underline{E} + \underline{B} \cdot \vec{v}_3) \quad (1.3.57)$$

Therefore

$$\underline{f} = q\underline{F} \cdot \vec{v} \quad (1.3.58)$$

relativizes both the electromagnetic power equation

$$P = q\underline{E} \cdot \vec{v}_3 \quad (1.3.59)$$

and the Lorentz force law

$$\underline{F} = q(\underline{E} + \underline{B} \cdot \vec{v}_3) \quad (1.3.60)$$

²The proper velocity, which is defined with respect to the proper time and has unit magnitude, is independent of the choice of time coordinate but requires a definition of length and so can not be defined topologically.