

1.4 Densities and volumes

1.4.1 Volume form and volume element

A **volume form** ϵ provides a measure or oriented unit density on a space and a (unit) **volume element** ϵ^{-1} provides an oriented unit volume. One induces the other via

$$\epsilon^{-1} \cdot \epsilon = 1 \quad (1.4.1)$$

For example, in three dimensions,

$$dV = \underline{\underline{\epsilon}} \cdot \underline{\underline{dV}} \quad (1.4.2)$$

is the volume of the infinitesimal volume element $\underline{\underline{dV}}$ and

$$\rho = \underline{\underline{\epsilon}} \cdot \underline{\underline{\rho}} \quad (1.4.3)$$

is the charge per unit volume of the charge density $\underline{\underline{\rho}}$. More generally, the volume form and element induce a **volume duality**¹ between n -vectors \mathbf{v} and $(N-n)$ -forms $\star\mathbf{v}$, and n -forms ω and $(N-n)$ -vectors $\star\omega$,

$$\begin{aligned} \star\mathbf{v} &\equiv \mathbf{v} \cdot \epsilon & , & & \mathbf{v} &= \epsilon^{-1} \cdot \star\mathbf{v} \\ \star\omega &\equiv \omega \cdot \epsilon^{-1} & , & & \omega &= \epsilon \cdot \star\omega \end{aligned} \quad (1.4.4)$$

Using Eq. (1.1.19),

$$\begin{aligned} \star^{-1}\mathbf{v} &= \epsilon \cdot \mathbf{v} = (-1)^{n(N-n)} \mathbf{v} \cdot \epsilon = (-1)^{n(N-n)} \star\mathbf{v} \\ \star^{-1}\omega &= \epsilon^{-1} \cdot \omega = (-1)^{n(N-n)} \omega \cdot \epsilon^{-1} = (-1)^{n(N-n)} \star\omega \end{aligned} \quad (1.4.5)$$

and so in three dimensions $\star^{-1} = \star$. For example, the traditional vector representation \vec{B} of the magnetic flux density $\underline{\underline{B}}$ is

$$\vec{B} = \star\underline{\underline{B}} \quad (1.4.6)$$

1.4.2 Divergence

We can use volume duality to define the divergence of an n -vector

$$(-1)^{n-1} \nabla \cdot \mathbf{v} \equiv \star^{-1} \nabla \wedge \star\mathbf{v} \quad (1.4.7)$$

See Figure 1.4.1 and Eq. (Q4.3.2).

For example, in three dimensions, the divergence of the traditional vector representation \vec{B} of the magnetic flux density $\underline{\underline{B}}$ is

$$\nabla \cdot \vec{B} = \star (\nabla \wedge \underline{\underline{B}}) \quad (1.4.8)$$

¹This duality is similar to the Hodge \star duality between n -forms and $(N-n)$ -forms defined in Section 2.1.3, so we use a similar notation.

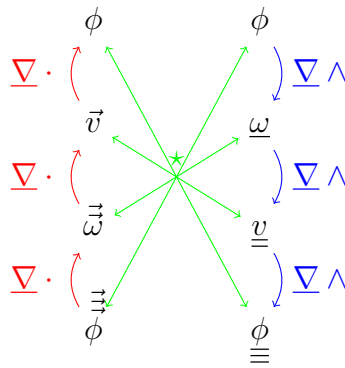


Figure 1.4.1: Volume duality in three dimensions.

1.4.3 Integration of multivectors

The volume form ϵ provides a measure for the integration of a scalar ϕ over an N -dimensional volume V

$$\int_V \phi \epsilon = \int_V \phi \epsilon \cdot d\mathbf{V} = \int_V \phi dV = \text{scalar} \tag{1.4.9}$$

More generally, we can use volume duality to integrate an $(N - n)$ -vector \mathbf{v} over an n -dimensional surface S

$$\begin{aligned} \int_S \star \mathbf{v} &= \int_S (\star \mathbf{v}) \cdot d\mathbf{S} = \int_S \mathbf{v} \cdot \star^{-1} d\mathbf{S} \\ &= \int_S \mathbf{v} \cdot \epsilon = \int_S (\mathbf{v} \cdot \epsilon) \cdot d\mathbf{S} = \int_S \mathbf{v} \cdot \epsilon \cdot d\mathbf{S} \\ &= \int_S (\mathbf{v} \cdot \mathbf{n}) \sigma = \int_S (\mathbf{v} \cdot \mathbf{n}) \sigma \cdot d\mathbf{S} = \int_S \mathbf{v} \cdot \mathbf{n} dS \end{aligned} \tag{1.4.10}$$

where \mathbf{n} is the normal $(N - n)$ -form to the surface S

$$\mathbf{n} \cdot d\mathbf{S} = 0 \tag{1.4.11}$$

σ is the volume form on S

$$\sigma \cdot d\mathbf{S} = dS \tag{1.4.12}$$

and we have used

$$\epsilon = \mathbf{n} \wedge \sigma \tag{1.4.13}$$

or equivalently

$$\star^{-1} d\mathbf{S} = \mathbf{n} dS \tag{1.4.14}$$

Combining Eqs. (1.2.9) and (1.4.7) gives the **divergence theorem**

$$(-1)^{n-1} \int_S \star (\nabla \cdot \mathbf{v}) = \int_{\partial S} \star \mathbf{v} \tag{1.4.15}$$

for an n -vector \mathbf{v} and an $(N - n + 1)$ -surface S .