Chapter 2

Differential geometry

2.1 Lengths and angles

2.1.1 Abstract index notation

The notation $\vec{v}$, $\underline{\omega}$, ... works well for multivectors and differential forms but is inadequate for more general tensors. Instead, we will use the **abstract index notation** in which a vector $\vec{v}$ is written $v^{\mathbf{a}}$ and a covector $\underline{\omega}$ is written $\omega_{\mathbf{a}}$

$$\vec{v} \;\leftrightarrow\; v^{\mathbf{a}} \quad , \quad \underline{\omega} \;\leftrightarrow\; \omega_{\mathbf{a}} \tag{2.1.1}$$

The abstract index $\mathbf{a}$ denotes the tensorial nature of the quantity by its position and does not take specific values. A contraction is denoted by repeated indices

$$\underline{\omega} \cdot \vec{v} \;\leftrightarrow\; \omega_{\mathbf{a}} v^{\mathbf{a}} \tag{2.1.2}$$

A **tensor** can have an arbitrary number of vector and covector indices $T^{\mathbf{ab}\cdots}_{\mathbf{cd}\cdots}$ and is expressed in components as

$$T^{\mathbf{ab}\cdots}_{\mathbf{cd}\cdots} = T^{\alpha\beta\cdots}_{\gamma\delta\cdots}\, e^{\mathbf{a}}_{\alpha} e^{\mathbf{b}}_{\beta} \ldots e^{\gamma}_{\mathbf{c}} e^{\delta}_{\mathbf{d}} \ldots \tag{2.1.3}$$

Note the difference between the two index notations. The abstract indices denote the tensorial nature of $T^{\mathbf{ab}\cdots}_{\mathbf{cd}\cdots}$, while the component indices label the components $T^{\alpha\beta\cdots}_{\gamma\delta\cdots}$ and basis vectors $\vec{e}_{\alpha}$ and covectors $\underline{e}^{\gamma}$, and are summed over in the above equation.

A tensor $T_{\mathbf{ab}}$ can be decomposed into **symmetric** and **antisymmetric** parts

$$T_{\mathbf{ab}} = T_{(\mathbf{ab})} + T_{[\mathbf{ab}]} \tag{2.1.4}$$

where round brackets denote the symmetric part

$$T_{(\mathbf{ab})} = \frac{1}{2}\left(T_{\mathbf{ab}} + T_{\mathbf{ba}}\right) \tag{2.1.5}$$

and square brackets denote the antisymmetric part

$$T_{[\mathbf{ab}]} = \frac{1}{2}\left(T_{\mathbf{ab}} - T_{\mathbf{ba}}\right) \tag{2.1.6}$$

An n-vector is an antisymmetric tensor with n vector indices and an n-form is an antisymmetric tensor with n covector indices, for example

$$\vec{\vec{v}} \;\leftrightarrow\; v^{[\mathbf{ab}]} \quad , \quad \underline{\underline{\omega}} \;\leftrightarrow\; \omega_{[\mathbf{ab}]} \tag{2.1.7}$$

A differential form can be expressed in components as a general tensor

$$\omega_{[\mathbf{a_1}\cdots\mathbf{a_n}]} = \omega_{\alpha_1\cdots\alpha_n} e^{\alpha_1}_{\mathbf{a_1}} \ldots e^{\alpha_n}_{\mathbf{a_n}} \tag{2.1.8}$$

but is more naturally expressed in terms of a differential form basis

$$\omega_{[\mathbf{a_1}\cdots\mathbf{a_n}]} = \frac{1}{n!}\omega_{\alpha_1\cdots\alpha_n}\, e^{\alpha_1}_{\mathbf{a_1}} \wedge \ldots \wedge e^{\alpha_n}_{\mathbf{a_n}} \tag{2.1.9}$$

and similarly for multivectors.

The contraction of a differential form with a multivector is

$$\begin{aligned} \underline{\underline{\omega}} \cdot \vec{\vec{v}} \;&\leftrightarrow\; \frac{1}{2!}\omega_{[\mathbf{ab}]} v^{[\mathbf{ab}]} \quad , \quad \underline{\underline{\underline{\omega}}} \cdot \vec{\vec{v}} \;\leftrightarrow\; \frac{1}{2!}\omega_{[\mathbf{abc}]} v^{[\mathbf{bc}]} \\ \underline{\underline{\omega}} \cdot \vec{\vec{\vec{v}}} \;&\leftrightarrow\; \frac{1}{2!}\omega_{[\mathbf{ab}]} v^{[\mathbf{abc}]} \quad , \quad \underline{\underline{\underline{\omega}}} \cdot \vec{\vec{\vec{v}}} \;\leftrightarrow\; \frac{1}{3!}\omega_{[\mathbf{abc}]} v^{[\mathbf{abc}]} \end{aligned} \tag{2.1.10}$$

and more generally the contraction of an m-form $\boldsymbol{\omega}$ with an n-vector $\boldsymbol{v}$ is

$$\boldsymbol{\omega} \cdot \boldsymbol{v} \;\leftrightarrow\; \begin{cases} \dfrac{1}{m!}\omega_{[\mathbf{b_1}\cdots\mathbf{b_m}]} v^{[\mathbf{b_1}\cdots\mathbf{b_m}\mathbf{a_{m+1}}\cdots\mathbf{a_n}]} & \text{for } m \leq n \\ \dfrac{1}{n!}\omega_{[\mathbf{a_{n+1}}\cdots\mathbf{a_m}\mathbf{b_1}\cdots\mathbf{b_n}]} v^{[\mathbf{b_1}\cdots\mathbf{b_n}]} & \text{for } m \geq n \end{cases} \tag{2.1.11}$$

The divergence of a multivector follows the same pattern

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} \;\leftrightarrow\; \nabla_{\mathbf{b}} v^{[\mathbf{b}\mathbf{a_2}\cdots\mathbf{a_n}]} \tag{2.1.12}$$

The exterior product of differential forms is

$$\underline{\omega} \wedge \underline{\sigma} \;\leftrightarrow\; \omega_{\mathbf{a}}\sigma_{\mathbf{b}} - \omega_{\mathbf{b}}\sigma_{\mathbf{a}} \tag{2.1.13}$$

$$\underline{\omega} \wedge \underline{\underline{\sigma}} \;\leftrightarrow\; \omega_{\mathbf{a}}\sigma_{[\mathbf{bc}]} + \omega_{\mathbf{b}}\sigma_{[\mathbf{ca}]} + \omega_{\mathbf{c}}\sigma_{[\mathbf{ab}]} \tag{2.1.14}$$

and more generally the exterior product of an m-form $\boldsymbol{\omega}$ with an n-form $\boldsymbol{\sigma}$ is

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} \;\leftrightarrow\; \frac{(m+n)!}{m!\,n!}\omega_{[\mathbf{a_1}\cdots\mathbf{a_m}}\sigma_{\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.15}$$

and similarly for multivectors. The exterior derivative follows the same pattern

$$\underline{\nabla} \wedge \phi \;\leftrightarrow\; \nabla_{\mathbf{a}}\phi \tag{2.1.16}$$

$$\underline{\nabla} \wedge \underline{\omega} \;\leftrightarrow\; \nabla_{\mathbf{a}}\omega_{\mathbf{b}} - \nabla_{\mathbf{b}}\omega_{\mathbf{a}} \tag{2.1.17}$$

$$\underline{\nabla} \wedge \underline{\underline{\omega}} \;\leftrightarrow\; \nabla_{\mathbf{a}}\omega_{[\mathbf{bc}]} + \nabla_{\mathbf{b}}\omega_{[\mathbf{ca}]} + \nabla_{\mathbf{c}}\omega_{[\mathbf{ab}]} \tag{2.1.18}$$

and more generally the exterior derivative of a n-form $\boldsymbol{\omega}$ is

$$\boldsymbol{\nabla} \wedge \boldsymbol{\omega} \;\leftrightarrow\; (n+1)\nabla_{[\mathbf{a}}\omega_{\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.19}$$

2.1.2 Metric

The **metric** $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ define lengths and angles in a space. They are symmetric tensors

$$g_{\mathbf{ab}} = g_{\mathbf{ba}} \quad , \quad g^{\mathbf{ab}} = g^{\mathbf{ba}} \tag{2.1.20}$$

and related by

$$g^{\mathbf{ab}} g_{\mathbf{bc}} = \delta^{\mathbf{a}}_{\mathbf{c}} \tag{2.1.21}$$

where the identity tensor $\delta^{\mathbf{a}}_{\mathbf{b}}$ has the property

$$\delta^{\mathbf{a}}_{\mathbf{b}} v^{\mathbf{b}} = v^{\mathbf{a}} \quad , \quad \delta^{\mathbf{b}}_{\mathbf{a}} \omega_{\mathbf{b}} = \omega_{\mathbf{a}} \tag{2.1.22}$$

and similarly for other tensors.

The metric gives the **inner product** of vectors and covectors

$$\vec{u} \cdot \vec{v} = g_{\mathbf{ab}} u^{\mathbf{a}} v^{\mathbf{b}} \quad , \quad \underline{\omega} \cdot \underline{\sigma} = g^{\mathbf{ab}} \omega_{\mathbf{a}} \sigma_{\mathbf{b}} \tag{2.1.23}$$

and more generally of n-forms

$$\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = \frac{1}{n!}\, g^{\mathbf{a_1 b_1}} \ldots g^{\mathbf{a_n b_n}}\, \omega_{[\mathbf{a_1 \cdots a_n}]} \sigma_{[\mathbf{b_1 \cdots b_n}]} \tag{2.1.24}$$

and similarly for n-vectors. Note that these inner products depend on the metric, in contrast to the contraction of an n-vector with an n-form. The inner product of two n-vectors or n-forms can be expressed as

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|\, |\boldsymbol{v}| \cos\theta \tag{2.1.25}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

The metric induces a **metric duality** $\diamond$ between vectors and covectors

$$\begin{aligned} v_{\mathbf{a}} &= g_{\mathbf{ab}} v^{\mathbf{b}} \quad , \quad v^{\mathbf{a}} = g^{\mathbf{ab}} v_{\mathbf{b}} \\ \omega^{\mathbf{a}} &= g^{\mathbf{ab}} \omega_{\mathbf{b}} \quad , \quad \omega_{\mathbf{a}} = g_{\mathbf{ab}} \omega^{\mathbf{b}} \end{aligned} \tag{2.1.26}$$

and more generally n-vectors and n-forms

$$v_{[\mathbf{a_1 \cdots a_n}]} = g_{\mathbf{a_1 b_1}} \ldots g_{\mathbf{a_n b_n}}\, v^{[\mathbf{b_1 \cdots b_n}]} \tag{2.1.27}$$

$$\omega^{[\mathbf{a_1 \cdots a_n}]} = g^{\mathbf{a_1 b_1}} \ldots g^{\mathbf{a_n b_n}}\, \omega_{[\mathbf{b_1 \cdots b_n}]} \tag{2.1.28}$$

For example, the traditional vector representation $\vec{E}$ of the electric field $\underline{E}$ is

$$E^{\mathbf{a}} = g^{\mathbf{ab}} E_{\mathbf{b}} \tag{2.1.29}$$

More generally, the metric can **raise** or **lower indices** on any tensor [1]

$$T^{\mathbf{a}}{}_{\mathbf{b}} = g_{\mathbf{bc}} T^{\mathbf{ac}} \tag{2.1.30}$$

[1] It is important to maintain the horizontal position of the indices if the tensor is not symmetric since $T^{\mathbf{a}}{}_{\mathbf{b}} = g_{\mathbf{bc}} T^{\mathbf{ac}} \neq g_{\mathbf{bc}} T^{\mathbf{ca}} = T_{\mathbf{b}}{}^{\mathbf{a}}$ if $T^{\mathbf{ac}} \neq T^{\mathbf{ca}}$.

The inner product of basis vectors is

$$\vec{e}_\alpha \cdot \vec{e}_\beta = g_{\mathbf{ab}} e^{\mathbf{a}}_\alpha e^{\mathbf{b}}_\beta = g_{\alpha\beta} \tag{2.1.31}$$

An **orthonormal basis** has metric components $g_{\alpha\beta} = \delta_{\alpha\beta}$. In a coordinate system x^α, the length ds of an infinitesimal displacement $dx^{\mathbf{a}}$ can be expressed as

$$ds^2 = g_{\mathbf{ab}} dx^{\mathbf{a}} dx^{\mathbf{b}} = g_{\alpha\beta} dx^\alpha dx^\beta \tag{2.1.32}$$

For example, in Cartesian coordinates in two dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 \tag{2.1.33}$$

and so the components of the metric are $g_{xx} = g_{yy} = 1$ and $g_{xy} = 0$. In polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{2.1.34}$$

and so $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and $g_{r\theta} = 0$. **Cartesian bases** are the only orthonormal coordinate bases and they exist only in flat spaces.

The identity tensor on the space of n-forms and n-vectors

$$\delta^{[\mathbf{a_1}\cdots\mathbf{a_n}]}_{[\mathbf{b_1}\cdots\mathbf{b_n}]} = n!\, \delta^{[\mathbf{a_1}}_{[\mathbf{b_1}} \cdots \delta^{\mathbf{a_n}]}_{\mathbf{b_n}]} \tag{2.1.35}$$

has the property

$$\frac{1}{n!} \delta^{[\mathbf{a_1}\cdots\mathbf{a_n}]}_{[\mathbf{b_1}\cdots\mathbf{b_n}]} v^{[\mathbf{b_1}\cdots\mathbf{b_n}]} = v^{[\mathbf{a_1}\cdots\mathbf{a_n}]} \quad , \quad \frac{1}{n!} \delta^{[\mathbf{b_1}\cdots\mathbf{b_n}]}_{[\mathbf{a_1}\cdots\mathbf{a_n}]} \omega_{[\mathbf{b_1}\cdots\mathbf{b_n}]} = \omega_{[\mathbf{a_1}\cdots\mathbf{a_n}]} \tag{2.1.36}$$

and can be expressed in components as

$$\begin{aligned}
\delta^{[\mathbf{a_1}\cdots\mathbf{a_n}]}_{[\mathbf{b_1}\cdots\mathbf{b_n}]} &= \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\, e^{\mathbf{a_1}}_{\alpha_1} \cdots e^{\mathbf{a_n}}_{\alpha_n}\, e^{\beta_1}_{\mathbf{b_1}} \cdots e^{\beta_n}_{\mathbf{b_n}} && (2.1.37) \\
&= \frac{1}{(n!)^2} \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} \left(e^{\mathbf{a_1}}_{\alpha_1} \wedge \ldots \wedge e^{\mathbf{a_n}}_{\alpha_n} \right) \left(e^{\beta_1}_{\mathbf{b_1}} \wedge \ldots \wedge e^{\beta_n}_{\mathbf{b_n}} \right) && (2.1.38) \\
&= \frac{1}{n!} \left(e^{\mathbf{a_1}}_{\alpha_1} \wedge \ldots \wedge e^{\mathbf{a_n}}_{\alpha_n} \right) \left(e^{\alpha_1}_{\mathbf{b_1}} \wedge \ldots \wedge e^{\alpha_n}_{\mathbf{b_n}} \right) && (2.1.39)
\end{aligned}$$

The metric on the space of n-forms and n-vectors is

$$g_{[\mathbf{a_1}\cdots\mathbf{a_n}][\mathbf{c_1}\cdots\mathbf{c_n}]} = g_{\mathbf{a_1 b_1}} \cdots g_{\mathbf{a_n b_n}}\, \delta^{[\mathbf{b_1}\cdots\mathbf{b_n}]}_{[\mathbf{c_1}\cdots\mathbf{c_n}]} \tag{2.1.40}$$

The metric and inverse metric are related by

$$\frac{1}{n!} g^{[\mathbf{a_1}\cdots\mathbf{a_n}][\mathbf{b_1}\cdots\mathbf{b_n}]} g_{[\mathbf{b_1}\cdots\mathbf{b_n}][\mathbf{c_1}\cdots\mathbf{c_n}]} = \delta^{[\mathbf{a_1}\cdots\mathbf{a_n}]}_{[\mathbf{c_1}\cdots\mathbf{c_n}]} \tag{2.1.41}$$

The inner product of two n-forms is

$$\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = \frac{1}{(n!)^2} g^{[\mathbf{a_1}\cdots\mathbf{a_n}][\mathbf{b_1}\cdots\mathbf{b_n}]} \omega_{[\mathbf{a_1}\cdots\mathbf{a_n}]} \sigma_{[\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.42}$$

and similarly for n-vectors, and metric duality between n-vectors and n-forms is

$$v_{[\mathbf{a_1}\cdots\mathbf{a_n}]} = \frac{1}{n!} g_{[\mathbf{a_1}\cdots\mathbf{a_n}][\mathbf{b_1}\cdots\mathbf{b_n}]} v^{[\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.43}$$

$$\omega^{[\mathbf{a_1}\cdots\mathbf{a_n}]} = \frac{1}{n!} g^{[\mathbf{a_1}\cdots\mathbf{a_n}][\mathbf{b_1}\cdots\mathbf{b_n}]} \omega_{[\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.44}$$

2.1.3 Metric and volume form

The volume form determines volumes and orientations but not lengths or angles, while the metric determines lengths and angles, and hence volumes, but not orientations. Thus the volume form is determined up to its orientation by the metric. Explicitly, Eq. (1.4.1) generalizes to

$$\epsilon^{-1[\mathbf{a_1}\cdots\mathbf{a_N}]}\epsilon_{[\mathbf{b_1}\cdots\mathbf{b_N}]} = \delta^{[\mathbf{a_1}\cdots\mathbf{a_N}]}_{[\mathbf{b_1}\cdots\mathbf{b_N}]} \tag{2.1.45}$$

and lowering indices gives

$$\epsilon^{-1}_{[\mathbf{a_1}\cdots\mathbf{a_N}]}\epsilon_{[\mathbf{b_1}\cdots\mathbf{b_N}]} = g_{[\mathbf{a_1}\cdots\mathbf{a_N}][\mathbf{b_1}\cdots\mathbf{b_N}]} \tag{2.1.46}$$

therefore

$$\epsilon_{[\mathbf{a_1}\cdots\mathbf{a_N}]}\epsilon_{[\mathbf{b_1}\cdots\mathbf{b_N}]} = (\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon})\, g_{[\mathbf{a_1}\cdots\mathbf{a_N}][\mathbf{b_1}\cdots\mathbf{b_N}]} \tag{2.1.47}$$

where $\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} = \pm 1$ is needed because of sign differences between $\epsilon_{[\mathbf{a_1}\cdots\mathbf{a_N}]}$ and $\epsilon^{-1}_{[\mathbf{a_1}\cdots\mathbf{a_N}]}$ that can arise in spacetimes. Now, taking components of Eqs. (2.1.40) and (2.1.47), the determinant of the metric components

$$g = g_{1\cdots N 1\cdots N} = \frac{\epsilon^2_{1\cdots N}}{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}} \tag{2.1.48}$$

and so

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} = \operatorname{sgn} g \tag{2.1.49}$$

and

$$\epsilon_{1\cdots N} = \sqrt{|g|} \tag{2.1.50}$$

Contracting Eq. (2.1.45) gives

$$\frac{1}{(N-n)!}\epsilon^{-1[\mathbf{a_1}\cdots\mathbf{a_n}\mathbf{c_{n+1}}\cdots\mathbf{c_N}]}\epsilon_{[\mathbf{b_1}\cdots\mathbf{b_n}\mathbf{c_{n+1}}\cdots\mathbf{c_N}]} = \delta^{[\mathbf{a_1}\cdots\mathbf{a_n}]}_{[\mathbf{b_1}\cdots\mathbf{b_n}]} \tag{2.1.51}$$

corresponding to the Levi-Civita Kronecker delta relation.

Hodge duality

Combining metric duality $\diamond$ between n-forms and n-vectors with volume duality $\star$ between n-vectors and $(N-n)$-forms, see Section 1.4.1, gives a mapping between n-forms and $(N-n)$-forms called the **Hodge dual**

$$* = \star\diamond \tag{2.1.52}$$

For example, the tensors $\underline{\underline{\vec{\varepsilon}_0}}$ and $\underline{\vec{\vec{\mu}}_0}^{-1}$ in Eqs. (1.3.2) and (1.3.3) are

$$\varepsilon_0{}^{\mathbf{c}}_{[\mathbf{ab}]} = \varepsilon_0\epsilon_{[\mathbf{abd}]}g^{\mathbf{dc}} \tag{2.1.53}$$

$$\mu_0^{-1}{}^{[\mathbf{bc}]}_{\mathbf{a}} = \mu_0^{-1}\epsilon_{[\mathbf{ade}]}g^{\mathbf{db}}g^{\mathbf{ec}} \tag{2.1.54}$$

so, in natural units in which $\varepsilon_0 = \mu_0^{-1} = 1$, Eqs. (1.3.2) and (1.3.3) become

$$\underline{\underline{D}} = *\underline{E} + \underline{\underline{P}} \tag{2.1.55}$$

$$\underline{H} = *\underline{\underline{B}} - \underline{M} \tag{2.1.56}$$

Also, following Eq. (1.4.7), the divergence of an n-form $\boldsymbol{\omega}$ is [2]

$$\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \diamond \boldsymbol{\nabla} \cdot \diamond \boldsymbol{\omega} = (-1)^{n-1} *^{-1} \boldsymbol{\nabla} \wedge * \boldsymbol{\omega} \tag{2.1.57}$$

In three dimensions, the volume $\star$, metric $\diamond$ and Hodge $*$ dualities can be used to map any antisymmetric tensor into either a scalar or a vector, see Figure 2.1.1. This is why

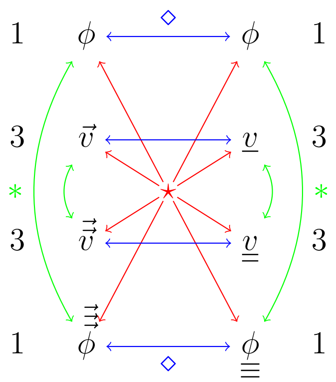

Figure 2.1.1: In three dimensions, the volume $\star$, metric $\diamond$ and Hodge $*$ dualities can be used to map any antisymmetric tensor into either a scalar or a vector.

traditional vector calculus works in three dimensions. However, this is not true in four or more dimensions. In particular, in four dimensions Hodge duality maps two-forms non-trivially into two-forms.

[2] The mathematical notation for $\boldsymbol{\nabla} \cdot \boldsymbol{\omega}$ is $-\delta\omega$ where δ is the codifferential.