Random Walk in 2D Square Lattice

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Randomness - What and How

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Randomness - What and How

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How do we get a Random Walk?

1. DRINK as much as you can
Randomness - What and How

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- How do we get a Random Walk?
  1. *DRINK* as much as you can
  2. Just Walk without any thought
2D Random Walk

- a Typical Random Walk in Statistical Physics
  * 1-D Picture *
2D Random Walk

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  * 1-D Picture *
- Get More Realistic
2D Random Walk

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- Still too complicated? - Make use of computer!
Overview of the plan

1. Make 10,000 trial walks with 1,000 steps for each (in C)
2. Visualize one trajectory to see if it works well
3. Show the distribution for the final positions
4. Interpret the distribution (with more data)
5. Briefly verify the result
a 1,000-step Random Walk (gnuplot)
Distribution for 10,000 such trials
Average Distance $D$ v.s. Number of Steps $N$

- $\log D = a \log N + b$ (Leaner Fitting in Gnuplot)
Average Distance $D$ v.s. Number of Steps $N$

- $\log D = a \log N + b$ (Leaner Fitting in Gnuplot)
  - $a = 0.54 \pm 0.048$ and $b = -0.3 \pm 0.4$
Explicit Calculation for 1D Case

- Probability, in a total of $N$ steps, of making $n$ steps to the right is
  \[ W(n) = \frac{N!}{n!(N-n)!} p^n (1 - p)^{N-n} \]

  where $p$ is the probability of stepping right.
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- Some statistical properties of Binomial Distribution \(W(n)\)
  - Mean value \(< n >= Np\)
  - Dispersion \(< (\Delta n)^2 >= Npq\)
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  - Mean value $<n> = Np$
  - Dispersion $<(\Delta n)^2> = Npq$

- The average displacement $m$ satisfies,
  - $m = n - (N - n) = 2n - N$
  - $<m> = 2 <n> - N = N(2p - 1) = 0$
  - $<(\Delta m)^2> = <(2\Delta n)^2> = 4Npq = N$ so that $\sigma_m = \sqrt{N}$
Intuitive View . . . general case inc. 2D square lattice

- **Final Displacement**

\[ \vec{l} = \sum_{i=1}^{N} \vec{l}_i \]

Therefore, \( \sigma_l = \sqrt{N} \) as long as each step is random and symmetric.
Intuitive View ... general case inc. 2D square lattice

- Final Displacement
  \[ \vec{l} = \sum_{i=1}^{N} \vec{l}_i \]

- Mean value
  \[ \langle \vec{l} \rangle = \sum_{i=1}^{N} \langle \vec{l}_i \rangle = 0 \]
Intuitive View . . . general case inc. 2D square lattice

- **Final Displacement**

  \[ \mathbf{\vec{l}} = \sum_{i=1}^{N} \mathbf{\vec{l}}_i \]

- **Mean value**

  \[ < \mathbf{\vec{l}} > = \sum_{i=1}^{N} < \mathbf{\vec{l}}_i > = 0 \]

- **Dispersion**

  \[ < \mathbf{\vec{l}}^2 > = < \left( \sum_{i=1}^{N} \mathbf{\vec{l}}_i \right) \left( \sum_{i=1}^{N} \mathbf{\vec{l}}_i \right) > = \sum_{i=1}^{N} < \mathbf{\vec{l}}_i^2 > + \sum_{i,j=1}^{N} < \mathbf{\vec{l}}_i \cdot \mathbf{\vec{l}}_j > = N \]

  and hence \( \sigma_l = \sqrt{N} \) as long as each step is *Random* and *Symmetric*. 
Comparison with Our Data

\[ \log D = a \log N + b, \text{ i.e. } D = e^b N^a \]
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  a = 0.54 \pm 0.048 \\
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THANK YOU FOR YOUR ATTENTION