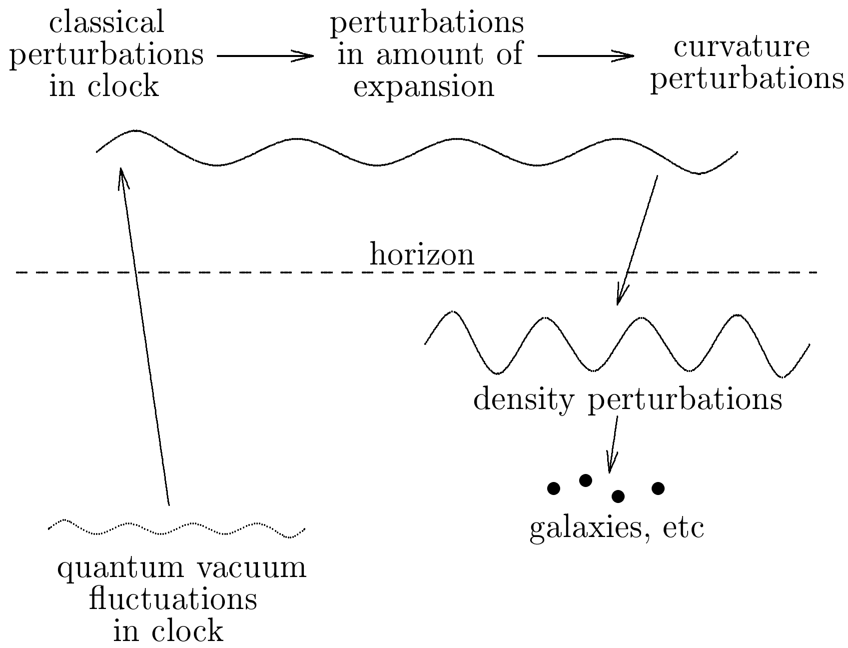
3.5 Generating perturbations

The general mechanism for the generation of perturbations during inflation is sketched in the figure. We will first consider the simplest case of slow-roll inflation with a single component inflaton, and then go on to discuss the more general formulation.



3.5.1 Single component inflaton

In Section 3.4 we neglected the perturbations in the metric. Here, we must include them as they are what we will be trying to calculate.

To first order in perturbation theory, the scalar, vector and tensor perturbations decouple from each other. We will focus on the scalar perturbations because they eventually become the density perturbations which grow to form galaxies and the all the rest of the large scale structure in the universe. The tensor perturbations, which correspond to gravitational waves, are in principle also interesting but in practice probably have an unobservably small amplitude. The vector perturbations decay and so are not likely to be interesting.

In the case of a single component inflaton, the action is

$$S = \int \left[-\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right] \sqrt{-g}\, d^4x \qquad (197)$$

There are many different gauge invariant variables we could choose to represent the scalar perturbations. The best choice is

$$\varphi \equiv a \left(\delta\phi - \frac{\dot{\phi}}{H}\mathcal{R} \right) \qquad (198)$$

which is a times the scalar field perturbation on spatially flat hypersurfaces ($-\frac{2}{3}\frac{1}{a^2}\nabla^2\mathcal{R}$ is the spatial curvature perturbation). Once the perturbations leave the horizon, we will want to reinterpret this variable in terms of

$$\mathcal{R}_\mathrm{c} = -\left(\frac{H}{a\dot{\phi}}\right)\varphi = \mathcal{R} - \frac{H}{\dot{\phi}}\delta\phi = \mathcal{R} + H\left(v + B\right) \qquad (199)$$

which is the curvature perturbation on constant ϕ or comoving ($v + B = 0$) hypersurfaces. $\mathcal{R}_\mathrm{c}$ is a convenient quantity because it is constant on superhorizon scales in a universe containing just a single component of matter (see Section 3.5.2 for clarification and qualification of this statement).

A somewhat lengthy but straightforward calculation gives the action for the scalar perturbations

$$S = \int \frac{1}{2} \left[(\varphi')^2 - (\boldsymbol{\nabla}\varphi)^2 + \frac{H}{a\dot{\phi}} \left(\frac{a\dot{\phi}}{H} \right)'' \varphi^2 \right] d\eta \, d^3\mathbf{x} \tag{200}$$

where a prime denotes the derivative with respect to conformal time η. Note that this includes the metric perturbations coming from both the gravitational and scalar field parts of the action. The equation of motion is

$$\varphi'' - \nabla^2\varphi - \frac{H}{a\dot{\phi}} \left(\frac{a\dot{\phi}}{H} \right)'' \varphi = 0 \tag{201}$$

This has the general solution

$$\varphi(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_{\mathbf{k}} \, \varphi_k(\eta) + a^\dagger_{-\mathbf{k}} \, \varphi^*_k(\eta) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{202}$$

where φ_k satisfies

$$\varphi''_k + k^2\varphi_k - \frac{H}{a\dot{\phi}} \left(\frac{a\dot{\phi}}{H} \right)'' \varphi_k = 0 \tag{203}$$

and is normalized such that

$$\varphi_k \varphi^{*\prime}_k - \varphi'_k \varphi^*_k = i \tag{204}$$

The quantization condition

$$[\varphi(\eta, \mathbf{x}), \varphi'(\eta, \mathbf{y})] = i \, \delta^3(\mathbf{x} - \mathbf{y}) \tag{205}$$

gives

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}} \right] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{206}$$

On small scales we have

$$\varphi''_k + k^2\varphi_k = 0 \tag{207}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{208}$$

so

$$a_{\mathbf{k}}|0\rangle = 0 \tag{209}$$

corresponds to the usual flat space vacuum on small scales.

On large scales we have

$$\varphi''_k - \frac{H}{a\dot{\phi}} \left(\frac{a\dot{\phi}}{H} \right)'' \varphi_k = 0 \tag{210}$$

which has solution

$$\varphi_k = A_k \frac{a\dot{\phi}}{H} + B_k \frac{a\dot{\phi}}{H} \int \left(\frac{H}{a\dot{\phi}} \right)^2 d\eta \tag{211}$$

where A_k and B_k are constants. The growing mode is

$$\varphi_k = A_k \frac{a\dot{\phi}}{H} \tag{212}$$

Note that φ_k and φ^*_k have the same time dependence.

This allows us to rewrite the large-scale Fourier modes as

$$a_{\mathbf{k}}\,\varphi_k(\eta) + a^\dagger_{-\mathbf{k}}\,\varphi^*_k(\eta) = b_{\mathbf{k}}\frac{a\dot\phi}{H} \tag{213}$$

where

$$b_{\mathbf{k}} = A_k a_{\mathbf{k}} + A^*_k a^\dagger_{-\mathbf{k}} \tag{214}$$

Now

$$\left[b_{\mathbf{k}}, b^\dagger_{\mathbf{l}}\right] = 0 \tag{215}$$

and so the large-scale Fourier modes are **classical** Gaussian random variables with

$$\langle 0|b_{\mathbf{k}}b^\dagger_{\mathbf{l}}|0\rangle = |A_k|^2\,\delta^3(\mathbf{k}-\mathbf{l}) \tag{216}$$

From Eqs. (199) and (213)

$$\mathcal{R}_{\rm c}(\mathbf{k},\eta) = -b_{\mathbf{k}} \tag{217}$$

The $\mathcal{R}_{\rm c}(\mathbf{k},\eta)$ are thus constant, independent, Gaussian magnitude, random phase, classical random variables, and are determined entirely by their **power spectrum**, $P_{\mathcal{R}_{\rm c}}(k)$, which is defined by

$$\langle\mathcal{R}_{\rm c}(\mathbf{k},\eta)\mathcal{R}^*_{\rm c}(\mathbf{l},\eta)\rangle = \frac{2\pi^2}{k^3}P_{\mathcal{R}_{\rm c}}\delta^3(\mathbf{k}-\mathbf{l}) \tag{218}$$

The normalization is chosen so that

$$\langle\mathcal{R}_{\rm c}(\mathbf{x},\eta)\mathcal{R}_{\rm c}(\mathbf{y},\eta)\rangle = \int\frac{dk}{k}P_{\mathcal{R}_{\rm c}}\frac{\sin\left(k\,|\mathbf{x}-\mathbf{y}|\right)}{k\,|\mathbf{x}-\mathbf{y}|} \tag{219}$$

Using Eq. (216), we have

$$P_{\mathcal{R}_{\rm c}}(k) = \frac{k^3}{2\pi^2}\,|A_k|^2 \tag{220}$$

To get our final answer, we need to determine A_k by matching the long wavelength solution, Eq. (212), to the short wavelength solution, Eq. (208). In **slow-roll** inflation, H and $\dot\phi$ are slowly varying, and so we can match the short and long wavelength solutions using an approximate solution which treats H and $\dot\phi$ as constants during horizon crossing. For H and $\dot\phi$ constant, $\eta = -1/(aH)$ and Eq. (203) becomes

$$\varphi''_k + k^2\varphi_k - \frac{2}{\eta^2}\varphi_k = 0 \tag{221}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)e^{-ik\eta} \to \frac{i}{\sqrt{2k}}\frac{aH}{k} \qquad \text{as} \quad \frac{k}{aH}\to 0 \tag{222}$$

Matching this to Eq. (212) gives

$$A_k = \frac{i}{\sqrt{2k^3}}\frac{H^2}{\dot\phi} \tag{223}$$

Within our approximation, we can choose to evaluate the right hand side at any time around horizon crossing. For definiteness, we evaluate it at horizon crossing

$$A_k = \left.\frac{i}{\sqrt{2k^3}}\frac{H^2}{\dot\phi}\right|_{aH=k} \tag{224}$$

and so from Eq. (220)

$$P_{\mathcal{R}_{\rm c}}(k) = \left.\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot\phi}\right)^2\right|_{aH=k} \tag{225}$$

We can use the slow-roll approximation to rewrite this in terms of the potential and its derivatives

$$P_{\mathcal{R}_c}^{1/2}(k) = \frac{H}{2\pi}\frac{H}{|\dot{\phi}|} = \frac{1}{2\pi\sqrt{3}}\frac{V^{3/2}}{|V'|} \tag{226}$$

where, as before, the right-hand side should be evaluated at $aH = k$. A similar calculation for the tensor perturbations gives the spectrum of gravitational waves produced during slow-roll inflation

$$P_{\mathrm{T}}^{1/2}(k) = \frac{H}{2\pi} = \frac{V^{1/2}}{2\pi\sqrt{3}} \tag{227}$$

The **spectral index** is defined by

$$n = 1 + \frac{d\ln P}{d\ln k} \tag{228}$$

$n = 1$ corresponds to a scale-invariant spectrum. Using the slow-roll approximation, Eqs. (226) and (227) give

$$n_{\mathcal{R}_c} = 1 - 2\frac{\ddot{\phi}}{H\dot{\phi}} - 4\left(\frac{-\dot{H}}{H^2}\right) = 1 + 2\frac{V''}{V} - 3\left(\frac{V'}{V}\right)^2 \tag{229}$$

and

$$n_{\mathrm{T}} = 1 - 2\left(\frac{-\dot{H}}{H^2}\right) = 1 - \left(\frac{V'}{V}\right)^2 \tag{230}$$

Note that

$$1 - n_{\mathrm{T}} = \frac{P_{\mathrm{T}}}{P_{\mathcal{R}_c}} \tag{231}$$

for single component slow-roll inflation.

References

1. A. A. Starobinsky, Physics Letters B117 (1982) 175-178.
2. V. F. Mukhanov, Soviet Physics JETP 67 (1988) 1297-1302.
3. E. D. Stewart and D. H. Lyth, Physics Letters B302 (1993) 171-175.

3.5.2 General formulation

The general formulation is based on

$$\Delta\mathcal{R} = \delta N \tag{232}$$

This equation is valid on superhorizon scales, $k \ll aH$, if the anisotropic stress, π, is negligible. $\Delta\mathcal{R} = \mathcal{R}(t_2) - \mathcal{R}(t_1)$ is the change in $\mathcal{R}$ between some final hypersurface, t_2, and some initial hypersurface, t_1, and δN is the perturbation in the number of e-folds of expansion between the two hypersurfaces.

Taking t_1 to be a flat hypersurface, so $\mathcal{R}(t_1) = 0$, and t_2 to be a comoving hypersurface $(av - B = 0)$ gives

$$\mathcal{R}_{\rm c}(t_2) = \delta N \tag{233}$$

It is convenient to choose the final hypersurface to be comoving because $\mathcal{R}_{\rm c}$ becomes constant (on superhorizon scales) once the matter in the universe has effectively become just a single component, for example, when everything has decayed to radiation. We take t_2 to be some time after $\mathcal{R}_{\rm c}$ has become constant, i.e. after the matter in the universe has effectively become just a single component.

For a multi-component inflaton, ϕ^a, we can write Eq. (233) as

$$\mathcal{R}_{\rm c}(t_2) = \delta N = \frac{\partial N}{\partial \phi^a}\delta\phi^a_{\mathcal{R}}(t_1) \tag{234}$$

which is a precise mathematical statement of the superhorizon process illustrated in the figure at the beginning of this section.

The spectrum is given by

$$\begin{aligned}
\frac{2\pi^2}{k^3}P_{\mathcal{R}_{\rm c}}\delta^3(\mathbf{k}-\mathbf{l}) &= \langle \mathcal{R}_{\rm c}(\mathbf{k},t_2)\,\mathcal{R}^*_{\rm c}(\mathbf{l},t_2)\rangle & (235)\\
&= \left\langle \delta\phi^a_{\mathcal{R}}(\mathbf{k},t_1)\,\delta\phi^{b*}_{\mathcal{R}}(\mathbf{l},t_1)\right\rangle \frac{\partial N}{\partial\phi^a}\frac{\partial N}{\partial\phi^b} & (236)\\
&= \frac{H^2}{2k^3}\delta^3(\mathbf{k}-\mathbf{l})\,h^{ab}\frac{\partial N}{\partial\phi^a}\frac{\partial N}{\partial\phi^b} & (237)
\end{aligned}$$

where h_{ab} is the metric on the scalar field space. Therefore

$$P_{\mathcal{R}_{\rm c}} = \left(\frac{H}{2\pi}\right)^2 h^{ab}\frac{\partial N}{\partial\phi^a}\frac{\partial N}{\partial\phi^b} \tag{238}$$

For a single component inflaton,

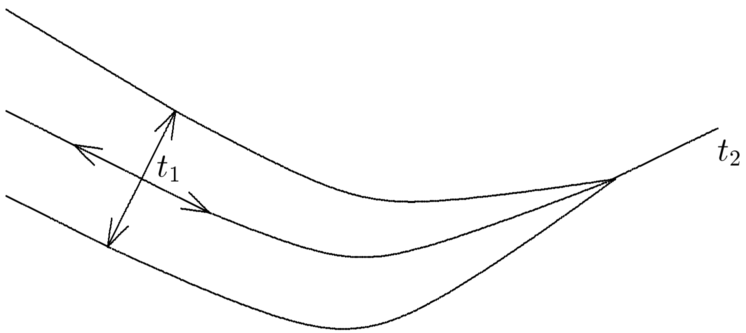
$$\frac{\partial N}{\partial\phi} = -\frac{H}{\dot\phi} \tag{239}$$

and so we recover Eq. (225), but now with a clear understanding of the meaning of the $H/\dot\phi$ factor.

For a multi-component inflaton,

$$\frac{\partial N}{\partial \phi^a}\dot{\phi}^a = -H \tag{240}$$

Thus, fluctuations along the trajectory of the inflaton give a contribution equal to that in the single component case. This contribution depends only on quantities evaluated at horizon crossing. This is because fluctuations along the trajectory do not change the trajectory and so only give a local contribution to δN.



However, fluctuations orthogonal to the trajectory of the inflaton kick the inflaton to a new trajectory and so change the entire history up until the time when the trajectories coalesce due to the matter in the universe becoming effectively just composed of a single component. Thus, fluctuations orthogonal to the trajectory give a non-local contribution to δN causing the perturbations to become sensitive to the whole of the history between t_1 and t_2.

Note that because the orthogonal fluctuations give an extra contribution to $P_{\mathcal{R}_c}$, Eq. (231) is modified to

$$1 - n_{\mathrm{T}} \geq \frac{P_{\mathrm{T}}}{P_{\mathcal{R}_c}} \tag{241}$$

References

1. A. A. Starobinsky, JETP Letters 42 (1985) 152-155.
2. M. Sasaki and E. D. Stewart, Progress of Theoretical Physics 95 (1996) 71-78.