3 Inflation

3.1 Motivation

In this section we will try to understand some of the basic observed properties of the universe listed in Section 1.2. For simplicity we will assume approximate homogeneity and isotropy.

5. & 1. **Expanding** and **old**
General relativity tells us that the universe is dynamical and so would be expected to be either expanding or contracting. From

$$3H^2 + 3K = \rho \tag{121}$$

we see that if $\rho \geq 0$ and $K \leq 0$ the universe will expand forever.

2. **Big**
An old expanding universe should be big, but how big? It would be natural to create a Planck size universe expanding at the Planck rate, $L \sim H \sim 1$. The current value of the Hubble parameter is $H_0 \sim 10^{-60}$, and so for a universe dominated by radiation or matter since the Planck epoch the current size of the universe would be $L_0 \sim 10^{30} \sim 0.1\,\text{mm}$ or $L_0 \sim 10^{40} \sim 10^3\,\text{km}$ respectively. We know the universe is bigger: $L_0 \gtrsim 1/H_0 \sim 10^{60}$. To start with $LH \sim 1$ and end up with $LH \gtrsim 1$ we need

$$\frac{d}{dt}(LH) \geq 0\,, \quad \text{i.e.} \quad \ddot{a} \geq 0 \tag{122}$$

3. **A lot of matter**
The universe contains a lot of matter, $M_0 \gtrsim 10^{60}$. Where did it all come from? To create matter or energy in an expanding universe requires $p < 0$. [1] However, we don't just need to create energy, we need to create energy rapidly enough to get $E_0 \gtrsim \rho_0 (1/H_0)^3 \sim \rho_0^{-1/2}$. It is natural to expect the universe to be created with a Planck mass of energy, $E \sim 1$, at the Planck density, $\rho \sim 1$. To go from $E \sim \rho^{-1/2}$ to $E \gtrsim \rho^{-1/2}$ requires

$$\frac{d\ln E}{d\ln a} \geq -\frac{1}{2}\frac{d\ln\rho}{d\ln a}\,, \quad \text{i.e.} \quad \frac{d\ln E}{d\ln a} \geq 1 \tag{123}$$

This requires $p \leq -\rho/3$ which gives $\ddot{a} \geq 0$.

6. **No observable spatial curvature**
We know that the spatial curvature is now smaller than the energy density, $|K_0| < \rho_0 \sim 10^{-120}$. The initial conditions at the Planck epoch ($\rho \sim 1$) required to achieve this in a universe dominated by radiation or matter are $|K| \lesssim 10^{-60}$ or $|K| \lesssim 10^{-40}$ respectively. Even at the time of nucleosynthesis it requires $|K| \lesssim 10^{-15}\rho$. These can hardly be regarded as sensible initial conditions. Instead it would be natural to create a universe with $|K| \sim \rho \sim 1$. To evolve from $|K| \sim \rho$ to $|K| < \rho$ requires

$$-\frac{d\ln\rho}{d\ln a} \leq -\frac{d\ln K}{d\ln a} = 2\,, \quad \text{i.e.} \quad \ddot{a} \geq 0 \tag{124}$$

[1] Negative gravitational potential energy is generated at the same time so the total energy is conserved.

Thus, the fact that the universe is big, $L_0 \gtrsim 1/H_0$, that it contains a lot of matter, $M_0 \gtrsim 1/H_0$, and that it has no observable spatial curvature, $|K_0| < H_0^2$, all suggest that

$$\ddot{a} \geq 0 \tag{125}$$

on average during the history of the universe. We know that much of the recent expansion of the universe occurred with $\ddot{a} < 0$, and so require a sufficiently long earlier epoch with $\ddot{a} > 0$.

4. **Homogeneous** and **isotropic**
   We are assuming approximate homogeneity and isotropy and so all we can hope to explain is how to make an approximately homogeneous and isotropic universe more homogeneous and isotropic on the appropriate scales.

   If $\ddot{a} > 0$, comoving scales leave the horizon. Therefore comoving inhomogeneities and anisotropies will get stretched beyond the horizon. $\ddot{a} > 0$ also implies that ρ decreases more slowly than a^{-2}. Therefore discrete inhomogeneities will get diluted. Thus if $\ddot{a} > 0$, scales fixed relative to the horizon will tend to become homogeneous and isotropic.

   Note that any unwanted relics (particles, black holes, topological defects) from the early universe can be viewed as inhomogeneities and are got rid of in the same way.

7. **Density perturbations**
   If $\ddot{a} > 0$, vacuum (or even thermal) fluctuations on subhorizon scales can be magnified into classical perturbations on superhorizon scales. This will be the subject of Section 3.6.

References

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2. A. H. Guth, Physical Review D23 (1981) 347-356.

3. A. D. Linde, Physics Letters B108 (1982) 389-393; A. Albrecht and P. J. Steinhardt, Physical Review Letters 48 (1982) 1220-1223.

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3.2 Definition of inflation

We have seen that many of the basic observed properties of the universe can be explained by a sufficiently long epoch with $\ddot{a} > 0$. This is the usual definition of inflation.

Inflation is characterized by

Repulsive gravity

$$\ddot{a} > 0 \tag{126}$$

Comoving scales leave the horizon

$$\frac{d}{dt}\left(\frac{\lambda_{\rm phys}}{1/H}\right) > 0 \tag{127}$$

Sufficiently slowly decreasing Hubble parameter

$$-\frac{d\ln H}{d\ln a} < 1 \tag{128}$$

Curvature decreases relative to the energy density

$$-\frac{d\ln K}{d\ln a} > -\frac{d\ln\rho}{d\ln a} \tag{129}$$

Sufficiently negative pressure

$$p < -\frac{1}{3}\rho \tag{130}$$

The amount of inflation is measured by

$$\mathcal{N} = \ln\dot{a} = \ln(aH) \tag{131}$$

During inflation, H is usually approximately constant, and so $\mathcal{N}$ is approximately equivalent to the number of e-folds of expansion

$$N = \ln a = \int H dt \tag{132}$$

which is what is usually used to describe the time during inflation.

It is useful to know when a given comoving scale k crosses the horizon during inflation. Defining horizon crossing to occur when $aH = k$, a comoving scale crosses the horizon a number of e-folds N before the end of inflation given by

$$N = N_{\rm end} - N_{\rm cross} = \mathcal{N}_{\rm end} - \mathcal{N}_{\rm cross} + \ln\left(\frac{H_{\rm cross}}{H_{\rm end}}\right) \tag{133}$$

and

$$\begin{aligned}\mathcal{N}_{\rm end} - \mathcal{N}_{\rm cross} &= (\mathcal{N}_{\rm end} - \mathcal{N}_{\rm nuc}) + (\mathcal{N}_{\rm nuc} - \mathcal{N}_0) - (\mathcal{N}_{\rm cross} - \mathcal{N}_0) \\ &= (\mathcal{N}_{\rm end} - \mathcal{N}_{\rm nuc}) + 20 - \ln\left(\frac{k}{a_0 H_0}\right)\end{aligned} \tag{134}$$

Subscript nuc denotes the beginning of nucleosynthesis, which is taken to be when the temperature was $T = 10\,\mathrm{MeV}$. This is the earliest time at which the evolution of the universe is well understood and observationally tested. What happened before nucleosynthesis is highly speculative. $\mathcal{N}_{\rm end} - \mathcal{N}_{\rm nuc}$ could be positive or negative, with positive values restricted to be $\lesssim 40$. The current scales of observational cosmology span the range $\ln(k/a_0 H_0) \sim 0$ to 15.