3.4 Quantum fields in de Sitter space

In this section we will investigate the behavior of a quantized scalar field in de Sitter space.

For simplicity, we will assume that the spacetime is homogeneous and isotropic despite the fact that the scalar field is not, i.e. we will neglect the back-reaction of the fluctuations in the scalar field on the metric. This will be consistent if the perturbations in the energy density, pressure, etc., are negligible. For example, the behavior of the scalar field near the maximum of the potential in rolling scalar field inflation (Section 3.3.4) can be described using this formalism.

In a homogeneous and isotropic expanding universe with metric

$$ds^2 = dt^2 - a(t)^2 \mathbf{dx}^2 \tag{162}$$

the action for a free massive real scalar field

$$S = \int \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - m^2 \phi^2 \right] \sqrt{-g} \, d^4 x \tag{163}$$

becomes

$$S = \int \frac{1}{2} \left[\dot{\phi}^2 - \frac{1}{a^2} \left(\boldsymbol{\nabla} \phi \right)^2 - m^2 \phi^2 \right] a^3 \, dt \, d^3\mathbf{x} \tag{164}$$

Introducing the conformal time η

$$d\eta = \frac{dt}{a} \tag{165}$$

which is defined to make the metric conformally flat

$$ds^2 = a(\eta)^2 \left[d\eta^2 - \mathbf{dx}^2 \right] \tag{166}$$

denoting the derivative with respect to η by a prime

$$\phi' = a\dot{\phi} \tag{167}$$

and defining

$$\varphi = a\phi \tag{168}$$

gives

$$S = \int \frac{1}{2} \left[\varphi'^2 - \left(\boldsymbol{\nabla} \varphi \right)^2 - \left(a^2 m^2 - \frac{a''}{a} \right) \varphi^2 - \left(\frac{a'}{a} \varphi^2 \right)' \right] d\eta \, d^3\mathbf{x} \tag{169}$$

The equation of motion is

$$\varphi'' - \nabla^2 \varphi + \left(a^2 m^2 - \frac{a''}{a} \right) \varphi = 0 \tag{170}$$

This has the general solution

$$\varphi(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_{\mathbf{k}} \, \varphi_k(\eta) + a^\dagger_{-\mathbf{k}} \, \varphi^*_k(\eta) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{171}$$

where φ_k satisfies

$$\varphi''_k + \left(k^2 + a^2 m^2 - \frac{a''}{a} \right) \varphi_k = 0 \tag{172}$$

and is normalized such that

$$\varphi_k \varphi^{*\prime}_k - \varphi'_k \varphi^*_k = i \tag{173}$$

The quantization condition

$$\left[\varphi(\eta, \mathbf{x}), \varphi'(\eta, \mathbf{y}) \right] = i \, \delta^3(\mathbf{x} - \mathbf{y}) \tag{174}$$

gives

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}} \right] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{175}$$

In de Sitter space, H is constant, $a = e^{Ht}$,

$$\eta = -\frac{1}{aH} \tag{176}$$

and Eq. (172) becomes

$$\varphi_k'' + k^2\varphi_k + \frac{1}{\eta^2}\left(\frac{m^2}{H^2} - 2\right)\varphi_k = 0 \tag{177}$$

On scales well inside the horizon, $-k\eta \to \infty$, this reduces to

$$\varphi_k'' + k^2\varphi_k = 0 \tag{178}$$

which has normalized solution

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(A_k e^{-ik\eta} + B_k e^{ik\eta}\right) , \quad |A_k|^2 - |B_k|^2 = 1 \tag{179}$$

If the inflationary expansion has been going on for sufficiently long, the scalar field should be in the usual flat space vacuum state on scales well inside the horizon. Therefore, we should take $B_k = 0$ so that $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ correspoond to the usual flat space annihilation and creation operators, and the state should be $|0\rangle$ where

$$a_{\mathbf{k}}|0\rangle = 0 \tag{180}$$

We are free to take $A_k = 1$ to get

$$\varphi_k \to \frac{1}{\sqrt{2k}} e^{-ik\eta} \qquad \text{as} \quad -k\eta \to \infty \tag{181}$$

The solution of Eq. (177) which matches onto Eq. (181) on scales well inside the horizon is

$$\varphi_k = e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}\sqrt{\frac{\pi}{4k}}\sqrt{-k\eta}\, H_\nu^{(1)}(-k\eta) \tag{182}$$

where

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \tag{183}$$

A useful special case is $m = 0$ which gives $\nu = \frac{3}{2}$ and

$$\varphi_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} \tag{184}$$

On scales well outside the horizon we have

$$\varphi_k \to e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}\left(\frac{2^\nu\Gamma(\nu)}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})}\right)\frac{1}{\sqrt{2k}}(-k\eta)^{\frac{1}{2}-\nu} \quad \text{as} \quad -k\eta \to 0 \tag{185}$$

If $m^2 \leq \frac{9}{4}H^2$ then $\nu \in \Re$ and so φ_k and φ_k^* have the same time dependence. This allows us to rewrite the superhorizon Fourier modes, i.e. those with $k \ll aH$, as

$$a_{\mathbf{k}}\,\varphi_k(\eta) + a_{-\mathbf{k}}^\dagger\,\varphi_k^*(\eta) = b_{\mathbf{k}}\left(\frac{2^\nu\Gamma(\nu)}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})}\right)\frac{1}{\sqrt{2k}}(-k\eta)^{\frac{1}{2}-\nu} \tag{186}$$

where

$$b_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}a_{\mathbf{k}} + e^{-i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}a_{-\mathbf{k}}^\dagger \tag{187}$$

Now

$$\left[b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger\right] = 0 \tag{188}$$

and so the superhorizon Fourier modes are **classical** Gaussian random variables with

$$\langle 0|b_{\mathbf{k}}b_{\mathbf{l}}^\dagger|0\rangle = \delta^3(\mathbf{k}-\mathbf{l}) \tag{189}$$

Now

$$\langle 0|\phi^2|0\rangle = \frac{1}{a^2}\langle 0|\varphi^2|0\rangle = \frac{1}{a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,|\varphi_k|^2 \tag{190}$$

Therefore, using Eq. (184), in the special case of $m = 0$

$$\begin{aligned}\langle 0|\phi^2|0\rangle &= \frac{1}{a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left(1+\frac{a^2H^2}{k^2}\right) \\ &= \frac{1}{(2\pi)^2}\int \frac{k\,dk}{a^2} + \left(\frac{H}{2\pi}\right)^2 \int \frac{dk}{k}\end{aligned} \tag{191}$$

The first term is the usual short wavelength divergence of quantum field theory and does not concern us here. The second term dominates on the classical superhorizon scales, $k \ll aH$, and gives a long wavelength divergence. We can understand the second term's meaning by restricting the range of integration to be from some fixed physical smoothing scale somewhat larger than the horizon, $k_2/a = \epsilon H$, to some fixed comoving long wavelength cutoff k_1 which crossed the scale k_2/a at $a = a_1$

$$\left(\frac{H}{2\pi}\right)^2 \int_{k_1=\epsilon a_1 H}^{k_2=\epsilon a H} \frac{dk}{k} = \left(\frac{H}{2\pi}\right)^2 \ln\frac{a}{a_1} \tag{192}$$

$\ln(a/a_1)$ is the number of e-folds of expansion N since $k_1 = k_2$. Therefore, the classical superhorizon contribution gives

$$\sqrt{\langle 0|\phi^2|0\rangle} = \frac{H}{2\pi}\sqrt{N} \tag{193}$$

corresponding to a random walk with step length $H/2\pi$ and number of steps N.

Introducing a small mass $m \ll H$ confines the random walk. Then

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2} \tag{194}$$

and, using Eqs. (190) and (185), the classical superhorizon contribution to $\langle 0|\phi^2|0\rangle$ is

$$\begin{aligned}\langle 0|\phi^2|0\rangle &\simeq \frac{1}{a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left(\frac{aH}{k}\right)^{2\nu-1} \\ &= \left(\frac{H}{2\pi}\right)^2 \int \frac{dk}{k}\left(\frac{aH}{k}\right)^{-\frac{2m^2}{3H^2}} \\ &= \frac{3H^4}{8\pi^2 m^2}\left[\left(\frac{k}{aH}\right)^{\frac{2m^2}{3H^2}}\right]\end{aligned} \tag{195}$$

Taking the limits of integration to be from $k \sim aH$ to $k = 0$, i.e. all the classical superhorizon scales, gives for $m^2 \ll H^2$

$$\langle 0|\phi^2|0\rangle \simeq \frac{3H^4}{8\pi^2 m^2} \tag{196}$$

References

1. Quantum Fields in Curved Space, N. D. Birrell and P. C. W. Davies, Cambridge University Press (1982, paperback 1984).