Homework 1 - Taylor expansion

A sufficiently smooth function $f(x)$ can be expanded in a **Taylor series** about a point x_0 as

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (x - x_0)^n \tag{H1.1}$$

where

$$a_n = f^{(n)}(x_0) \tag{H1.2}$$

and superscript (n) denotes the nth derivative. In physics, this provides a very useful approximation when $|x - x_0|$ is sufficiently small

$$f(x) = a_0 + a_1 (x - x_0) + \frac{1}{2} a_2 (x - x_0)^2 + \mathcal{O}\big((x - x_0)^3\big) \tag{H1.3}$$

with the first few terms usually being sufficient if the approximation is going to be useful.

Q1.1. By calculating their derivatives, determine the Taylor expansions of e^x, $\cos x$ and $\sin x$ about $x = 0$. Hence show that

$$e^{ix} = \cos x + i \sin x \tag{Q1.1.1}$$

where $i^2 = -1$.

A1.1. The derivatives of e^x are

$$\left(\frac{d}{dx}\right)^n e^x = e^x \tag{A1.1.1}$$

therefore, using Eqs. (H1.1) and (H1.2),

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{A1.1.2}$$

The derivatives of $\cos x$ are

$$\left(\frac{d}{dx}\right)^{2n} \cos x = (-1)^n \cos x \tag{A1.1.3}$$

$$\left(\frac{d}{dx}\right)^{2n+1} \cos x = (-1)^{n+1} \sin x \tag{A1.1.4}$$

therefore, using Eqs. (H1.1) and (H1.2),

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{A1.1.5}$$

The derivatives of $\sin x$ are

$$\left(\frac{d}{dx}\right)^{2n}\sin x = (-1)^n \sin x \quad \text{(A1.1.6)}$$

$$\left(\frac{d}{dx}\right)^{2n+1}\sin x = (-1)^n \cos x \quad \text{(A1.1.7)}$$

therefore, using Eqs. (H1.1) and (H1.2),

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{(A1.1.8)}$$

Therefore, using Eqs. (A1.1.2), (A1.1.5) and (A1.1.8),

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n \quad \text{(A1.1.9)}$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1} \quad \text{(A1.1.10)}$$

$$= \cos x + i \sin x \quad \text{(A1.1.11)}$$

Q1.2. A particle moving vertically above the surface of the Earth experiences a gravitational acceleration

$$\ddot{z} = -\frac{GM_\oplus}{(R_\oplus + z)^2} \quad \text{(Q1.2.1)}$$

where z is the height above the surface of the Earth, a dot denotes the time derivative, G is the gravitational constant, and $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth.

What is the gravitational acceleration near the surface of the Earth? Integrate Eq. (Q1.2.1) for motion near the surface of the Earth. How long would it take a particle launched vertically upwards at $100\,\mathrm{m\,s^{-1}}$ to return to the surface of the Earth? Estimate how long it would have to take the particle to return before the errors in your results become large.

A1.2. Near the surface of the Earth, $z \ll R_\oplus$, so the gravitational acceleration near the surface of the Earth is

$$\ddot{z} \simeq -\frac{GM_\oplus}{R_\oplus^2} \equiv -g \quad \text{(A1.2.1)}$$

$$\simeq -\frac{6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} \cdot 5.97\times10^{24}\,\mathrm{kg}}{(6.37\times10^6\,\mathrm{m})^2} \simeq -9.8\,\mathrm{m\,s^{-2}} \quad \text{(A1.2.2)}$$

Integrating gives

$$z \simeq z_0 + v_0 t - \frac{1}{2} g t^2 \quad \text{(A1.2.3)}$$

Initially, $t = 0$, $z = 0$ and $\dot{z} = 100\,\mathrm{m\,s^{-1}}$ so $z_0 = 0$ and $v_0 = 100\,\mathrm{m\,s^{-1}}$. Finally, $t = t_\mathrm{r}$ and $z = 0$ so

$$v_0 t_\mathrm{r} - \frac{1}{2} g t_\mathrm{r}^2 = 0 \tag{A1.2.4}$$

therefore it would return at

$$t_\mathrm{r} = \frac{2v_0}{g} \simeq \frac{2 \cdot 100\,\mathrm{m\,s^{-1}}}{9.8\,\mathrm{m\,s^{-2}}} \simeq 20\,\mathrm{s} \tag{A1.2.5}$$

The errors in Eq. (A1.2.1) become large when $z \gtrsim R_\oplus$. The maximum height occurs when $\dot{z} = 0$ and, from Eq. (A1.2.3),

$$\dot{z} = v_0 - gt \tag{A1.2.6}$$

therefore the maximum height occurs at a time

$$t_\mathrm{max} = \frac{v_0}{g} \tag{A1.2.7}$$

Therefore, using Eqs. (A1.2.3) and (A1.2.5), the maximum height is

$$z_\mathrm{max} = \frac{v_0^2}{2g} = \frac{g t_\mathrm{r}^2}{8} \tag{A1.2.8}$$

The errors will become large when $z_\mathrm{max} \gtrsim R_\oplus$, i.e. when

$$t_\mathrm{r} \gtrsim \sqrt{\frac{8R_\oplus}{g}} \sim 40\,\mathrm{min} \tag{A1.2.9}$$

Q1.3. Expand the relativistic energy of a massive particle

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{Q1.3.1}$$

in a Taylor series in $v \ll c$ to leading non-trivial order and hence derive the Newtonian energy of a particle. What is the error in the Newtonian formula for $v = 10\,\mathrm{km\,s^{-1}}$? For $v = c/2$?

A1.3. E is a function of v^2 and so its Taylor series will be a series in v^2. The derivatives of E with respect to v^2 are

$$\frac{dE}{d(v^2)} = \frac{m}{2\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \tag{A1.3.1}$$

$$\frac{d^2E}{(d(v^2))^2} = \frac{3m}{4c^2\left(1 - \frac{v^2}{c^2}\right)^{5/2}} \tag{A1.3.2}$$

therefore

$$E = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \mathcal{O}\left(\frac{mv^6}{c^4}\right) \tag{A1.3.3}$$

The first term diverges in the Newtonian limit $c \to \infty$ but is a constant and so can be subtracted off to give a sensible Newtonian definition of the energy of a particle

$$E_{\mathrm{N}} = \frac{1}{2}mv^2 \tag{A1.3.4}$$

There can be various measures of the error, but the relative error

$$\mathcal{E} = \frac{\text{approx} - \text{exact}}{\text{exact}} \tag{A1.3.5}$$

has the advantage of being dimensionless. The relative error in the Newtonian formula is

$$\mathcal{E}_{\mathrm{N}} = \frac{E_{\mathrm{N}} - (E - mc^2)}{E - mc^2} = -\frac{3}{4}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right) \tag{A1.3.6}$$

which for $v = 10\,\mathrm{km\,s^{-1}}$ is

$$\mathcal{E}_{\mathrm{N}} \simeq -\frac{3}{4}\left(\frac{10\,\mathrm{km\,s^{-1}}}{3.00 \times 10^5\,\mathrm{km\,s^{-1}}}\right)^2 \simeq -8 \times 10^{-10} \tag{A1.3.7}$$

and for $v = c/2$ is

$$\mathcal{E}_{\mathrm{N}} \simeq -0.2 \tag{A1.3.8}$$