Homework 5 - Empirical interactions

Q5.1. Calculate the speed in $\mathrm{km\,h^{-1}}$ which maximizes the safe flow of traffic. Assume all vehicles have length 4.5 m, driver reaction time 1 s, and coefficient of friction between tires and road 0.8. For the safe flow of traffic, the separation between vehicles should be at least the minimum distance in which a vehicle can stop.

A5.1. Let v be the speed of the vehicle, then the distance travelled in the reaction time is

$$x_{\mathrm{r}} = v \times 1\,\mathrm{s} \tag{A5.1.1}$$

Once the brakes are applied, we need to consider the deceleration due to the frictional force between the tires and the road. The maximal frictional force is given by Eq. (2.4.1)

$$F_{\mathrm{s}}^{\mathrm{max}} = 0.8\, F_{\mathrm{N}} \tag{A5.1.2}$$

and the normal force F_{N} between the tires and the road balances the gravitational force on the vehicle

$$F_{\mathrm{N}} = Mg \tag{A5.1.3}$$

where M is the mass of the vehicle and

$$g \simeq 9.8\,\mathrm{m\,s^{-2}} \tag{A5.1.4}$$

is the gravitational acceleration near the surface of the Earth, see Q1.2. Therefore the braking acceleration will be

$$\ddot{x}_{\mathrm{b}} = -\frac{F_{\mathrm{s}}^{\mathrm{max}}}{M} = -\,0.8\,g \tag{A5.1.5}$$

and so

$$\dot{x}_{\mathrm{b}}(t) = v - 0.8\,gt \tag{A5.1.6}$$

and

$$x_{\mathrm{b}}(t) = vt - \frac{1}{2}\,0.8\,gt^2 \tag{A5.1.7}$$

Setting $\dot{x}_{\mathrm{b}}(t_{\mathrm{b}}) = 0$ in Eq. (A5.1.6) gives the braking time

$$t_{\mathrm{b}} = \frac{v}{0.8\,g} \tag{A5.1.8}$$

and substituting this into Eq. (A5.1.7) gives the braking distance

$$x_{\mathrm{b}}(t_{\mathrm{b}}) = \frac{1}{2}\frac{v^2}{0.8\,g} \tag{A5.1.9}$$

Combining Eqs. (A5.1.1) and (A5.1.9) gives the stopping distance

$$x_{\mathrm{s}} = v\,\mathrm{s} + \frac{1}{2}\frac{v^2}{0.8\,g} \tag{A5.1.10}$$

Therefore, for the safe flow of traffic, the front of one vehicle should be separated from the front of the next vehicle by a distance

$$d = v\,\mathrm{s} + \frac{1}{2}\frac{v^2}{0.8\,g} + 4.5\,\mathrm{m} \tag{A5.1.11}$$

giving a traffic flow

$$f = \frac{v}{d} = \frac{v}{v\mathrm{s} + \frac{1}{2}\frac{v^2}{0.8g} + 4.5\,\mathrm{m}} \tag{A5.1.12}$$

therefore

$$\frac{df}{dv} = \frac{4.5\,\mathrm{m} - \frac{1}{2}\frac{v^2}{0.8g}}{\left(v\mathrm{s} + \frac{1}{2}\frac{v^2}{0.8g} + 4.5\,\mathrm{m}\right)^2} \tag{A5.1.13}$$

and so the safe traffic flow is maximized when

$$v = 3\sqrt{0.8\,g\,\mathrm{m}} = 8.4\,\mathrm{m\,s^{-1}} \simeq 30\,\mathrm{km\,h^{-1}} \tag{A5.1.14}$$

Q5.2. A ball of radius R is moving directly towards a wall with speed v, and is spinning with angular speed w in a plane perpendicular to the wall. The coefficient of kinetic friction between the ball and the wall is μ, and you can assume that the normal force acts elastically. Calculate the angle at which the ball bounces off the wall.

A5.2. Let the mass of the ball be M, then, using Eq. (2.3.46), the moment of inertia of the ball

$$I = \frac{2}{5}MR^2 \tag{A5.2.1}$$

Initially

$$v_\perp = v \tag{A5.2.2}$$
$$v_\parallel = 0 \tag{A5.2.3}$$
$$\dot{\theta} = \omega \tag{A5.2.4}$$

When the ball hits the wall, see Figure A5.2.1 left,

$$F_\parallel = \mu F_\perp \tag{A5.2.5}$$
$$\tau = R\mu F_\perp \tag{A5.2.6}$$

and so

$$\begin{aligned} \frac{dp_\parallel}{dt} &= \mu\frac{dp_\perp}{dt} \quad \therefore \quad \frac{dv_\parallel}{dt} = \mu\frac{dv_\perp}{dt} \\ \frac{dL}{dt} &= R\mu\frac{dp_\perp}{dt} \quad \therefore \quad \frac{d\dot{\theta}}{dt} = \frac{5\mu}{2R}\frac{dv_\perp}{dt} \end{aligned} \tag{A5.2.7}$$

This will continue until either the ball stops slipping and starts rolling along the wall, see Figure A5.2.1 center, or the ball leaves the wall, see Figure A5.2.1 right. We will consider each case separately.

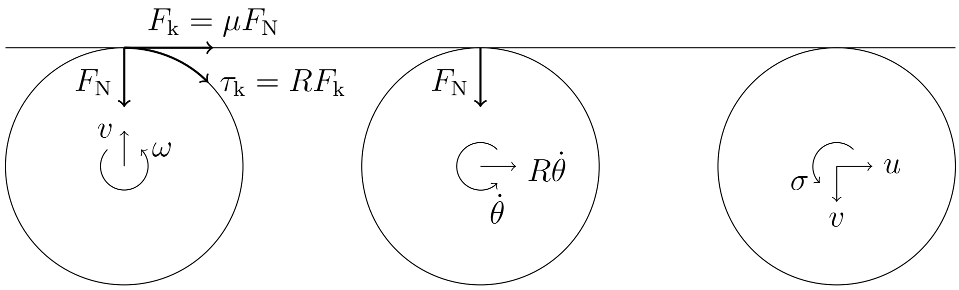

Figure A5.2.1: Left: initial state when the ball hits the wall; center: the ball stops slipping and starts rolling if its velocity parallel to the wall matches its radius times its angular velocity; right: final state when the ball leaves the wall.

Ball stops slipping before it leaves the wall The ball starts rolling when

$$v_\parallel = R\dot{\theta} \tag{A5.2.8}$$

and after which

$$F_\parallel = 0 \tag{A5.2.9}$$
$$\tau = 0 \tag{A5.2.10}$$

Therefore, using Eq. (A5.2.8),

$$u = R\sigma \tag{A5.2.11}$$

and, using Eq. (A5.2.7),

$$u = \frac{2R}{5}(\omega - \sigma) \tag{A5.2.12}$$

Solving gives

$$u = \frac{2R}{7}\omega \tag{A5.2.13}$$
$$\sigma = \frac{2}{7}\omega \tag{A5.2.14}$$

therefore the ball bounces back at an angle

$$\tan^{-1}\frac{u}{v} = \tan^{-1}\frac{2R\omega}{7v} \tag{A5.2.15}$$

from the perpendicular.

Ball leaves the wall before it stops slipping The ball leaves the wall when

$$v_\perp = -v \tag{A5.2.16}$$

due to the assumed elasticity of the normal force. Therefore, using Eq. (A5.2.7),

$$u = \mu\left[v - (-v)\right] \tag{A5.2.17}$$

and

$$\omega - \sigma = \frac{5\mu}{2R}\left[v - (-v)\right] \tag{A5.2.18}$$

Solving gives

$$u = 2\mu v \tag{A5.2.19}$$

$$\sigma = \omega - \frac{5\mu v}{R} \tag{A5.2.20}$$

therefore the ball bounces back at an angle

$$\tan^{-1}\frac{u}{v} = \tan^{-1} 2\mu \tag{A5.2.21}$$

from the perpendicular.

Summary The ball bounces back at an angle

$$\tan^{-1}\frac{u}{v} = \begin{cases} \tan^{-1}\dfrac{2R\omega}{7v} & \text{if } \omega \leq \dfrac{7\mu v}{R} \\ \tan^{-1} 2\mu & \text{if } \omega \geq \dfrac{7\mu v}{R} \end{cases} \tag{A5.2.22}$$

from the perpendicular.